

Marius Junge
Mathilde Perrin

THEORY OF \mathcal{H}_p -SPACES FOR CONTINUOUS
FILTRATIONS IN VON NEUMANN
ALGEBRAS

ASTÉRISQUE 362

Astérisque est un périodique de la Société mathématique de France.

Numéro 362, juin 2014

Comité de rédaction

Jean-Benoît BOST
Guy DAVID
Olivier DEBARRE
Damien GABORIAU
Guy HENNIART
Masaki KASHIWARA

Patrice LE CALVEZ
Guy MÉTIVIER
Robert Alan OLIVER
Raphaël ROUQUIER
Wolfgang SOERGEL
Wendelin WERNER

Yves ANDRÉ (dir.)

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 28 € (\$ 42)

Abonnement Europe : 530 € hors Europe : 569 € (\$ 853)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2014

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN 0303-1179

ISBN 978-2-85629-789-6

Directeur de la publication : Marc PEIGNÉ

THEORY OF \mathcal{H}_p -SPACES FOR CONTINUOUS
FILTRATIONS IN VON NEUMANN
ALGEBRAS

Marius Junge
Mathilde Perrin

M. Junge (*)

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801.

E-mail : `junge@math.uiuc.edu`

M. Perrin (**)

Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM,

Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera 13-15. 28049, Madrid, Spain.

E-mail : `mathilde.perrin@icmat.es`

2000 Mathematics Subject Classification. — Primary 46L53,46L52; secondary 46L51,60G44.

Key words and phrases. — Noncommutative L_p -spaces, noncommutative martingales, Hardy spaces, continuous filtration.

(*) Partially supported by National Science Foundation grants DMS-0901457 and DMS-1201886.

(**) Partially supported by the Agence Nationale de Recherche, ERC Grant StG-256997-CZOSQP and NSF grant DMS-1201886.

THEORY OF \mathcal{H}_p -SPACES FOR CONTINUOUS FILTRATIONS IN VON NEUMANN ALGEBRAS

Marius Junge, Mathilde Perrin

Abstract. — We introduce Hardy spaces for martingales with respect to continuous filtration for von Neumann algebras. In particular we prove the analogues of the Burkholder-Gundy and Burkholder-Rosenthal inequalities in this setting. The usual arguments using stopping times in the commutative case are replaced by tools from noncommutative function theory and allow us to obtain the analogue of the Feffermann-Stein duality and prove a noncommutative Davis decomposition.

Résumé (Théorie des espaces \mathcal{H}_p pour des filtrations continues dans des algèbres de von Neumann)

Nous introduisons des espaces de Hardy pour des martingales relatives à des filtrations continues d'algèbres de von Neumann. Nous démontrons en particulier les inégalités de Burkholder-Gundy et de Burkholder-Rosenthal dans ce cadre. Les arguments usuels basés sur des temps d'arrêt dans le cas commutatif sont remplacés par des outils de la théorie des fonctions non commutatives, qui nous permettent d'obtenir l'analogue de la dualité de Fefferman-Stein et de prouver une décomposition de Davis non commutative.

CONTENTS

1. Introduction	1
2. Preliminaries	7
2.1. Noncommutative L_p -spaces and martingales with respect to continuous filtrations	7
2.2. Ultraproduct techniques	8
2.3. L_p \mathcal{M} -modules	12
2.4. Free Rosenthal inequalities	15
3. The \mathcal{H}_p^c-spaces	19
3.1. The discrete case	19
3.2. Definitions of $\widehat{\mathcal{H}}_p^c$ and \mathcal{H}_p^c	21
3.3. Monotonicity and convexity properties	23
3.4. The spaces $\widehat{\mathcal{H}}_p^c$ and \mathcal{H}_p^c coincide	24
3.5. Ultraproduct spaces and L_p -modules	29
3.6. Injectivity results	31
3.7. Complementation results	32
3.8. Fefferman-Stein duality	34
3.9. Interpolation	39
4. Burkholder-Gundy inequalities	41
4.1. Burkholder-Gundy inequalities for $1 < p < 2$	41
4.2. Randrianantoanina's result in the discrete case	43
4.3. Sums of Banach spaces	45
4.4. Burkholder-Gundy inequalities for $2 < p < \infty$	47
4.5. The space \mathcal{H}_1	48
4.6. Interpolation	49
5. The \mathfrak{h}_p^c-spaces	51
5.1. The discrete case	51
5.2. Definitions of $\widehat{\mathfrak{h}}_p^c$, \mathfrak{h}_p^c and basic properties	54
5.3. Monotonicity and convexity properties	55
5.4. The spaces $\widehat{\mathfrak{h}}_p^c$ and \mathfrak{h}_p^c coincide	57
5.5. Injectivity results	64
5.6. Fefferman-Stein duality	65
5.7. Interpolation	67
6. Davis and Burkholder-Rosenthal inequalities	69
6.1. The discrete case	69
6.2. Definition of diagonal spaces for $1 \leq p < 2$ and basic properties	82
6.3. Davis decomposition for $1 \leq p < 2$	84
6.4. Davis inequalities for $2 \leq p < \infty$	91
6.5. Burkholder-Rosenthal inequalities	96

Appendix	97
Bibliography	99

CHAPTER 1

INTRODUCTION

The theory of stochastic integrals and martingales with continuous time is a well-known theory with many applications. Quantum stochastic calculus is also well developed with applications reaching into fields such as quantum optics. In the setting of von Neumann algebras, many classical martingale inequalities have been reformulated for noncommutative martingales with respect to discrete filtrations, see *e.g.* [40], [27], [21], [30]. The aim of this paper is to study martingales with respect to continuous filtrations in von Neumann algebras. Our long term goal is to develop a satisfactory theory for semimartingales, including the convergence of the stochastic integrals. In the noncommutative setting, we cannot construct the stochastic integrals pathwise as in [9]. It is unimaginable to consider the path of a process of operators in a von Neumann algebra. However, it is well-known that in the classical case, the convergence of the stochastic integrals is closely related to the existence of the quadratic variation bracket $[\cdot, \cdot]$ via the formula

$$X_t Y_t = \int^t X_{s-} dY_s + \int^t Y_{s-} dX_s + [X, Y]_t.$$

Here the quadratic variation bracket can be characterized as the limit in probability of the following dyadic square functions

$$[X, Y]_t = X_0 Y_0 + \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n - 1} (X_{t \frac{k+1}{2^n}} - X_{t \frac{k}{2^n}})(Y_{t \frac{k+1}{2^n}} - Y_{t \frac{k}{2^n}}).$$

Hence we will first study this quadratic variation bracket in the setting of von Neumann algebras, and then deal with stochastic integrals in a forthcoming paper based on the theory developed here. More precisely, we will focus on the $L_{\frac{1}{2}p}$ -norm of this bracket by considering the Hardy spaces H_p defined in the classical case by the norm

$$\|x\|_{H_p} = \|[x, x]\|_{\frac{1}{2}p}^{\frac{1}{2}}.$$

This paper develops a theory of the Hardy spaces of noncommutative martingales with respect to a continuous filtration. One fundamental application is an interpolation theory for these noncommutative function spaces which has already found applications in the theory of semigroups (see *e.g.* [22]).

Let us consider a von Neumann algebra \mathcal{M} . For simplicity, we assume that \mathcal{M} is finite and equipped with a normal faithful normalized trace τ . Fortunately, the theory of noncommutative H_p -spaces is now very well understood in the discrete setting, *i.e.*, when dealing with an increasing sequence $(\mathcal{M}_n)_{n \geq 0}$ of von Neumann subalgebras of \mathcal{M} , whose union is weak*-dense in \mathcal{M} . We consider the associated conditional expectations $\mathcal{E}_n : \mathcal{M} \rightarrow \mathcal{M}_n$. In the noncommutative setting it is well-known that we always encounter two different objects, the row and column versions of the Hardy spaces:

$$\|x\|_{H_p^c} = \left\| \left(\sum_n |d_n(x)|^2 \right)^{\frac{1}{2}} \right\|_p \quad \text{and} \quad \|x\|_{H_p^r} = \left\| \left(\sum_n |d_n(x^*)|^2 \right)^{\frac{1}{2}} \right\|_p,$$

where $d_n(x) = \mathcal{E}_n(x) - \mathcal{E}_{n-1}(x)$. Here $\|x\|_p = (\tau(|x|^p))^{1/p}$ refers to the norm in the noncommutative L_p -space. The noncommutative Burkholder-Gundy inequalities from [40] say that with equivalent norms for $1 < p < \infty$,

$$(1.0.1) \quad L_p(\mathcal{M}) = H_p,$$

where the H_p -space is defined by

$$H_p = \begin{cases} H_p^c + H_p^r & \text{for } 1 \leq p < 2, \\ H_p^c \cap H_p^r & \text{for } 2 \leq p < \infty. \end{cases}$$

Following the commutative theory, we should expect to define the bracket $[x, x]$ for a martingale x and then define

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x, x]\|_{\frac{1}{2}p}^{\frac{1}{2}} \quad \text{and} \quad \|x\|_{\widehat{\mathcal{H}}_p^r} = \|[x^*, x^*]\|_{\frac{1}{2}p}^{\frac{1}{2}}.$$

Armed with the definition we may then attempt to prove (1.0.1) for a continuous filtration $(\mathcal{M}_t)_{t \geq 0}$. For simplicity, we assume that the continuous parameter set is given by the interval $[0, 1]$. We define a candidate for the noncommutative bracket following a nonstandard analysis approach. For a finite partition $\sigma = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ of the interval $[0, 1]$ and $x \in \mathcal{M}$, we consider the finite bracket

$$[x, x]_\sigma = \sum_{t \in \sigma} |d_t^\sigma(x)|^2,$$

where $d_t^\sigma(x) = \mathcal{E}_t(x) - \mathcal{E}_{t-(\sigma)}(x)$. Then for $p > 2$, (1.0.1) gives an *a priori* bound

$$\|[x, x]_\sigma\|_{\frac{1}{2}p}^{\frac{1}{2}} \leq \alpha_p \|x\|_p.$$

Hence, for a fixed ultrafilter \mathcal{U} refining the general net of finite partitions of $[0, 1]$, we may simply define

$$[x, x]_{\mathcal{U}} = w\text{-}L_{\frac{1}{2}p}\text{-}\lim_{\sigma, \mathcal{U}} [x, x]_\sigma.$$

In fact, in nonstandard analysis, the weak-limit corresponds to the standard part and is known to coincide with the classical definition of the bracket for commutative martingales. However, the norm is only lower semi-continuous with respect to the weak topology and we should not expect Burkholder/Gundy inequalities for continuous filtrations to be a simple consequence of the discrete theory of H_p -spaces. Yet, using the crucial observation that the $L_{\frac{1}{2}p}$ -norms of the discrete brackets $[x, x]_\sigma$ are monotonous up to a constant, we may show the following result.

Theorem 1.0.1. — *Let $1 \leq p < \infty$ and $x \in \mathcal{M}$. Then*

$$\|[x, x]_{\mathcal{U}}\|_{\frac{1}{2}p} \simeq \lim_{\sigma, \mathcal{U}} \|[x, x]_\sigma\|_{\frac{1}{2}p} \simeq \begin{cases} \sup_{\sigma} \|[x, x]_\sigma\|_{\frac{1}{2}p} & \text{for } 1 \leq p < 2, \\ \inf_{\sigma} \|[x, x]_\sigma\|_{\frac{1}{2}p} & \text{for } 2 \leq p < \infty. \end{cases}$$

In particular, this implies that the $L_{\frac{1}{2}p}$ -norm of the bracket $[x, x]_{\mathcal{U}}$ does not depend on the choice of the ultrafilter \mathcal{U} , up to equivalent norm. We will discuss the independence of the bracket $[x, x]_{\mathcal{U}}$ itself from the choice of \mathcal{U} in a forthcoming paper. Hence for $1 \leq p < \infty$ and $x \in \mathcal{M}$ we define the norms

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x, x]_{\mathcal{U}}\|_{\frac{1}{2}p}^{\frac{1}{2}} \quad \text{and} \quad \|x\|_{\mathcal{H}_p^c} = \lim_{\sigma, \mathcal{U}} \|[x, x]_\sigma\|_{\frac{1}{2}p}^{\frac{1}{2}} = \lim_{\sigma, \mathcal{U}} \|x\|_{H_p^c(\sigma)}.$$

We denote by $\widehat{\mathcal{H}}_p^c$ and \mathcal{H}_p^c respectively the corresponding completions. Using theorem 1.0.1 we may show that actually

$$(1.0.2) \quad \widehat{\mathcal{H}}_p^c = \mathcal{H}_p^c \quad \text{with equivalent norms for } 1 \leq p < \infty.$$

Hence this defines a good candidate for the Hardy space of noncommutative martingales with respect to the continuous filtration $(\mathcal{M}_t)_{0 \leq t \leq 1}$. We now want to establish for this space the analogues of many well-known results in the discrete setting. For doing this, we will use the definition of the space \mathcal{H}_p^c , which will be more practical to work with. In particular, we may embed \mathcal{H}_p^c into some ultraproduct space, which has an L_p -module structure and a p -equiintegrability property. This