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HOLOMORPHIC ACTIONS, KUMMER EXAMPLES, AND ZIMMER PROGRAM

BY SERGE CANTAT AND ABDELGHANI ZEGHIB

ABSTRACT. – We classify compact Kähler manifolds M of dimension $n \geq 3$ on which acts a lattice of an almost simple real Lie group of rank $\geq n-1$. This provides a new line in the so-called Zimmer program, and characterizes certain complex tori as compact Kähler manifolds with large automorphisms groups.

RÉSUMÉ. – Nous classons les variétés compactes kählériennes M de dimension $n \geq 3$ munies d'une action d'un réseau Γ dans un groupe de Lie réel presque simple de rang $\geq n-1$. Ceci complète le programme de Zimmer dans ce cadre, et caractérise certains tores complexes compacts par des propriétés de leur groupe d'automorphismes.

1. Introduction

1.1. Zimmer Program

Let G be an almost simple real Lie group. The *real rank* $\mathsf{rk}_{\mathbf{R}}(G)$ of G is the dimension of a maximal Abelian subgroup of G that acts by $\mathbf{R}$-diagonalizable endomorphisms in the adjoint representation of G on its Lie algebra $\mathfrak{g}$; for example, the real Lie groups $\mathsf{SL}_n(\mathbf{R})$ and $\mathsf{SL}_n(\mathbf{C})$ have rank $n-1$. When $\mathsf{rk}_{\mathbf{R}}(G)$ is at least 2, we say that G is a *higher rank* almost simple Lie group. Let Γ be a *lattice* in a higher rank Lie group G; by definition, Γ is a discrete subgroup of G such that G/Γ has finite Haar volume. Margulis superrigidity theorem implies that all finite dimensional linear representations of Γ are built from representations in unitary groups and representations of the Lie group G itself. In particular, there is no faithful linear representation of Γ in dimension $\leq \mathsf{rk}_{\mathbf{R}}(G)$ (see Remark 3.4 below).

Zimmer's program predicts a similar picture for actions of Γ by diffeomorphisms on compact manifolds, at least when the dimension $\dim(V)$ of the manifold V is close to the minimal dimension of non trivial linear representations of G (see [27]). For instance, a central conjecture asserts that lattices in simple Lie groups of rank n do not act faithfully on compact manifolds of dimension less than n (see [64, 63, 65, 31]).

In this article, we pursue the study of Zimmer's program for holomorphic actions on compact Kähler manifolds, as initiated in [16] and [19, 20].

1.2. Automorphisms

Let M be a compact complex manifold of dimension n. By definition, diffeomorphisms of M which are holomorphic are called *automorphisms*. According to Bochner and Montgomery [11, 14], the group $\mathsf{Aut}(M)$ of all automorphisms of M is a complex Lie group, its Lie algebra is the algebra of holomorphic vector fields on M. Let $\mathsf{Aut}(M)^0$ be the connected component of the identity in $\mathsf{Aut}(M)$, and

$$\mathsf{Aut}(M)^\sharp = \mathsf{Aut}(M)/\mathsf{Aut}(M)^0$$

be the group of connected components. This group can be infinite, and is hard to describe: For example, it is not known whether there exists a compact complex manifold M for which $\mathsf{Aut}(M)^\sharp$ is not finitely generated.

When M is a Kähler manifold, Lieberman and Fujiki proved that $\mathsf{Aut}(M)^0$ has finite index in the kernel of the action of $\mathsf{Aut}(M)$ on the cohomology of M (see [28, 46]). Thus, if a subgroup Γ of $\mathsf{Aut}(M)$ embeds into $\mathsf{Aut}(M)^\sharp$, the action of Γ on the cohomology of M has finite kernel; in particular, the group $\mathsf{Aut}(M)^\sharp$ almost embeds in the group $\mathrm{Mod}(M)$ of isotopy classes of smooth diffeomorphisms of M. When M is simply connected or, more generally, has nilpotent fundamental group, $\mathrm{Mod}(M)$ is naturally described as the group of integer matrices in a linear algebraic group (see [56]). Thus, $\mathsf{Aut}(M)^\sharp$ sits naturally in an arithmetic lattice. Our main result goes in the other direction: It describes the largest possible lattices contained in $\mathsf{Aut}(M)^\sharp$.

1.3. Rigidity and Kummer examples

The main examples that provide large groups $\Gamma \subset \mathsf{Aut}(M)^\sharp$ come from linear actions on carefully chosen complex tori.

Example 1.1. – Let $E = \mathbf{C}/\Lambda$ be an elliptic curve and n be a positive integer. Let T be the torus $E^n = \mathbf{C}^n/\Lambda^n$. The group $\mathsf{Aut}(T)$ is the semi-direct product of $\mathsf{SL}(n, \mathsf{End}(E))$ by T, acting by translations on itself. In particular, the connected component $\mathsf{Aut}(T)^0$ coincides with the group of translations, and $\mathsf{Aut}(T)$ contains all linear transformations $z \mapsto B(z)$ where B is in $\mathsf{SL}_n(\mathbf{Z})$. If Λ is the lattice of integers $\mathcal{O}_d$ in an imaginary quadratic number field $\mathbf{Q}(\sqrt{d})$, where d is a squarefree negative integer, then $\mathsf{Aut}(T)$ contains a copy of $\mathsf{SL}_n(\mathcal{O}_d)$.

Example 1.2. – Starting with the previous example, one can change Γ into a finite index subgroup Γ_0, and change T into a quotient T/F where F is a finite subgroup of $\mathsf{Aut}(T)$ which is normalized by Γ_0. In general, T/F is an orbifold (a compact manifold with quotient singularities), and one needs to resolve the singularities in order to get an action on a smooth manifold M. The second operation that can be done is blowing up finite orbits of Γ. This provides infinitely many compact Kähler manifolds of dimension n with actions of lattices $\Gamma \subset \mathsf{SL}_n(\mathbf{R})$ (resp. $\Gamma \subset \mathsf{SL}_n(\mathbf{C})$).

In these examples, the group Γ is a lattice in a real Lie group of rank $(n-1)$, namely $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and Γ acts on a manifold M of dimension n. Moreover, the action of Γ on the cohomology of M has finite kernel and a finite index subgroup of Γ embeds in $\mathsf{Aut}(M)^\sharp$. Since this kind of construction is at the heart of the article, we introduce the following definition, which is taken from [17, 19].

DEFINITION 1.3. – *Let Γ be a group, and $\rho : \Gamma \to \mathsf{Aut}(M)$ a morphism into the group of automorphisms of a compact complex manifold M. This morphism is a* Kummer example *(or, equivalently, is of* Kummer type*) if there exists*

- *a birational morphism $\pi : M \to M_0$ onto an orbifold M_0,*
- *a finite orbifold cover $\epsilon : T \to M_0$ of M_0 by a torus T, and*
- *morphisms $\eta : \Gamma \to \mathsf{Aut}(T)$ and $\eta_0 : \Gamma \to \mathsf{Aut}(M_0)$*

such that $\epsilon \circ \eta(\gamma) = \eta_0(\gamma) \circ \epsilon$ and $\eta_0(\gamma) \circ \pi = \pi \circ \rho(\gamma)$ for all γ in Γ.

The notion of *orbifold* used in this text refers to compact complex analytic spaces with a finite number of singularities of quotient type; in other words, M_0 is locally the quotient of $(\mathbf{C}^n, 0)$ by a finite group of linear transformations (see Section 2.3).

Since automorphisms of a torus $\mathbf{C}^n/\Lambda$ are covered by affine transformations of $\mathbf{C}^n$, all Kummer examples come from actions of affine transformations on affine spaces.

1.4. Results

The following statement is our main theorem. It confirms Zimmer's program, in its strongest versions, for holomorphic actions on compact Kähler manifolds: We get a precise description of all possible actions of lattices $\Gamma \subset G$ for $\mathsf{rk}_{\mathbf{R}}(G) \geq \dim_{\mathbf{C}}(M) - 1$.

MAIN THEOREM. – *Let G be an almost simple real Lie group and Γ be a lattice in G. Let M be a compact Kähler manifold of dimension $n \geq 3$. Let $\rho : \Gamma \to \mathsf{Aut}(M)$ be a morphism with infinite image.*

(0) *The real rank $\mathsf{rk}_{\mathbf{R}}(G)$ is at most equal to the complex dimension of M.*

(1) *If $\mathsf{rk}_{\mathbf{R}}(G) = \dim(M)$, the group G is locally isomorphic to $\mathsf{SL}_{n+1}(\mathbf{R})$ or $\mathsf{SL}_{n+1}(\mathbf{C})$ and M is biholomorphic to the projective space $\mathbb{P}^n(\mathbf{C})$.*

(2) *If $\mathsf{rk}_{\mathbf{R}}(G) = \dim(M) - 1$, there exists a finite index subgroup Γ_0 in Γ such that either*

(2-a) *$\rho(\Gamma_0)$ is contained in $\mathsf{Aut}(M)^0$, and $\mathsf{Aut}(M)^0$ contains a subgroup which is locally isomorphic to G, or*

(2-b) *G is locally isomorphic to $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and the morphism $\rho : \Gamma_0 \to \mathsf{Aut}(M)$ is a Kummer example.*

Moreover, all examples corresponding to assertion (2-a) are described in Section 4.6 and all Kummer examples of assertion (2-b) are described in Section 7. In particular, for these Kummer examples, the complex torus T associated to M and the lattice Γ fall in one of the following three possible examples:

– $\Gamma \subset \mathsf{SL}_n(\mathbf{R})$ is commensurable to $\mathsf{SL}_n(\mathbf{Z})$ and T is isogenous to the product of n copies of an elliptic curve $\mathbf{C}/\Lambda$;

– $\Gamma \subset \mathsf{SL}_n(\mathbf{C})$ is commensurable to $\mathsf{SL}_n(\mathcal{O}_d)$ where $\mathcal{O}_d$ is the ring of integers in $\mathbf{Q}(\sqrt{d})$ for some negative integer d, and T is isogenous to the product of n copies of the elliptic curve $\mathbf{C}/\mathcal{O}_d$;

– In the third example, $n = 2k$ is even. There are positive integers a and b such that the quaternion algebra $\mathbf{H}_{a,b}$ over the rational numbers $\mathbf{Q}$ defined by the basis $(1, \mathsf{i}, \mathsf{j}, \mathsf{k})$, with

$$\mathsf{i}^2 = a,\ \mathsf{j}^2 = b,\ \mathsf{ij} = \mathsf{k} = -\mathsf{ji}$$

is an indefinite quaternion algebra and the lattice Γ is commensurable to the lattice $\mathsf{SL}_k(H_{a,b}(\mathbf{Z}))$. The torus T is isogenous to the product of k copies of an Abelian surface Y which contains $\mathbf{H}_{a,b}(\mathbf{Q})$ in its endomorphism algebra $\mathsf{End}_{\mathbf{Q}}(Y)$. Once a and b are fixed, those surfaces Y depend on one complex parameter, hence T depends also on one parameter; for some parameters Y is isogenous to the product of 2 copies of the elliptic curve $\mathbf{C}/\mathcal{O}_d$, and T is isogenous to $(\mathbf{C}/\mathcal{O}_d)^n$ with $d = -ab$ (see §7 for precise definitions and details).

As a consequence, Γ *is not cocompact, T is an Abelian variety and M is projective.*

Remark 1.4. – In dimension 2, [18] shows that all faithful actions of infinite discrete groups with Kazhdan property (T) by birational transformations on projective surfaces are birationally conjugate to actions by automorphisms on the projective plane $\mathbb{P}^2(\mathbf{C})$; thus, part (1) of the Main Theorem holds in the more general setting of birational actions and groups with Kazhdan property (T). Part (2) does not hold in dimension 2 for lattices in the rank 1 Lie group $\mathsf{SO}_{1,n}(\mathbf{R})$ (see [18, 26] for examples).

1.5. Strategy of the proof and complements

Sections 2 and 3 contain important preliminary facts, as well as a side result which shows how representation theory and Hodge theory can be used together in our setting (see §3.4).

The proof of the Main Theorem starts in §4: Assertion (1) is proved, and a complete list of all possible pairs (M, G) that appear in assertion (2-a) is obtained. This makes use of a previous result on Zimmer conjectures in the holomorphic setting (see [16], in which assertion (0) is proved), and classification of homogeneous or quasi-homogeneous spaces (see [2, 32, 40]). On our way, we describe Γ-invariant analytic subsets $Y \subset M$ and show that these subsets can be blown down to quotient singularities.

The core of the paper is to prove assertion (2-b) when the image $\rho(\Gamma_0)$ is not contained in $\mathsf{Aut}(M)^0$ (for all finite index subgroups of Γ) and $\mathsf{rk}_{\mathbf{R}}(G)$ is equal to $\dim(M) - 1$.

In that case, Γ acts almost faithfully on the cohomology of M, and this linear representation extends to a continuous representation of G on $H^*(M, \mathbf{R})$ (see §3). Section 5 shows that G preserves a non-trivial cone contained in the closure of the Kähler cone $\mathcal{K}(M) \subset H^{1,1}(M, \mathbf{R})$; this general fact holds for all linear representations of semi-simple Lie groups G for which a lattice $\Gamma \subset G$ preserves a salient cone. Section 5 can be skipped in a first reading.

Then, in §6, we apply ideas of Dinh and Sibony, of Zhang, and of our previous manuscripts together with representation theory. We fix a maximal torus A in G and study the eigenvectors of A in the G-invariant cone: Hodge index Theorem constrains the set of weights and eigenvectors; since the Chern classes are invariant under the action of G, this provides strong constraints on them. When there is no Γ-invariant analytic subset of positive dimension, Yau's Theorem can then be used to prove that M is a torus. To conclude the proof, we blow down all invariant analytic subsets to quotient singularities (see §4), and apply Hodge and Yau's Theorems in the orbifold setting.