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GAUSSIAN MEASURES ASSOCIATED TO THE HIGHER ORDER CONSERVATION LAWS OF THE BENJAMIN-ONO EQUATION

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ABSTRACT. – Inspired by the work of Zhidkov on the KdV equation, we perform a construction of weighted Gaussian measures associated to the higher order conservation laws of the Benjamin-Ono equation. The resulting measures are supported by Sobolev spaces of increasing regularity. We also prove a property on the support of these measures leading to the conjecture that they are indeed invariant by the flow of the Benjamin-Ono equation.

RÉSUMÉ. – Inspirés par le travail de Zhidkov sur l'équation KdV, nous construisons des mesures gaussiennes à poids associées à une loi de conservation arbitraire de l'équation de Benjamin-Ono. Les supports de ces mesures sont constitués de fonctions de régularité de Sobolev croissantes. On démontre aussi une propriété-clé des mesures qui nous conduit à conjecturer leur invariance par le flot de l'équation.

1. Introduction and statement of the results

1.1. Measures construction

The main goal of this article is to construct weighted Gaussian measures associated with an arbitrary conservation law of the Benjamin-Ono equation (BO), and thus to extend the result of the first author [14] which deals only with the first conservation law. The analysis contains several significant elaborations with respect to [14]; it requires an understanding of the interplay between the structure of the conservation laws of the Benjamin-Ono equation and the probabilistic arguments involved in the renormalization procedure defining the measures.

Let us recall that just like the KdV equation, the Benjamin-Ono equation is a basic dispersive PDE describing the propagation of one directional, long, small amplitude waves. The difference between the KdV and BO equations is that the KdV equation describes surface waves while the Benjamin-Ono equation models the propagation of internal waves. These models have rich mathematical structure from both the algebraic and analytical viewpoints.

In particular they have an infinite sequence of conservation laws. These aspects will be heavily exploited in the present work.

Consider now the Benjamin-Ono equation

$$\partial_t u + H\partial_x^2 u + u\partial_x u = 0, \tag{1.1}$$

with periodic boundary conditions (for simplicity throughout the paper we fix the period to be equal to 2π). In (1.1), H denotes the Hilbert transform acting on periodic distributions. Thanks to the work of Molinet [11] (1.1) is globally well-posed in H^s, $s \geq 0$ (see [13, 7, 5] for related results in the case when (1.1) is posed on the real line).

It is well-known that (smooth) solutions to (1.1) satisfy an infinite number of conservation laws (see e.g., [10, 1]). More precisely for $k \geq 0$ an integer, there is a conservation law of (1.1) of the form

$$E_{k/2}(u) = \|u\|^2_{\dot{H}^{k/2}} + R_{k/2}(u) \tag{1.2}$$

where $\dot{H}^s$ denotes the homogeneous Sobolev norm on periodic functions, and all the terms that appear in $R_{k/2}$ are homogeneous of the order larger than or equal to three in u. In Section 2, we will describe in more details the structure of $R_{k/2}$ for large k. Next we explicitly write the conservation laws $E_{k/2}$ for $k = 0, 1, 2, 3, 4$:

$$\begin{aligned}
E_0(u) &= \|u\|^2_{L^2};\\
E_{1/2}(u) &= \|u\|^2_{\dot{H}^{1/2}} + \frac{1}{3}\int u^3 dx;\\
E_1(u) &= \|u\|^2_{\dot{H}^1} + \frac{3}{4}\int u^2 H(u_x)dx + \frac{1}{8}\int u^4 dx;\\
E_{3/2}(u) &= \|u\|^2_{\dot{H}^{3/2}} - \int [\frac{3}{2}u(u_x)^2 + \frac{1}{2}uH(u_x)^2]dx\\
&\quad - \int [\frac{1}{3}u^3 H(u_x) + \frac{1}{4}u^2 H(uu_x)]dx - \frac{1}{20}\int u^5 dx;\\
E_2(u) &= \|u\|^2_{\dot{H}^2} - \frac{5}{4}\int [(u_x)^2 Hu_x + 2uu_{xx}Hu_x]dx\\
&\quad + \frac{5}{16}\int [5u^2(u_x)^2 + u^2 H(u_x)^2 + 2uH(\partial_x u)H(uu_x)]dx\\
&\quad + \int [\frac{5}{32}u^4 H(u_x) + \frac{5}{24}u^3 H(uu_x)]dx + \frac{1}{48}\int u^6 dx
\end{aligned}$$

where $\int$ is understood as the integral on the period $(0, 2\pi)$.

Following the work by Zhidkov [15] (see also [2, 8]), one may try to define an invariant measure for (1.1) by re-normalizing the formal measure $\exp(-E_{k/2}(u))du$. This re-normalization is a delicate procedure. One possibility would be first to re-normalize $\exp(-\|u\|^2_{\dot{H}^{k/2}})du$ as a Gaussian measure on an infinite dimensional space and then to show that the factor $\exp(-R_{k/2}(u))$ is integrable with respect to this measure.

Since $\exp(-\|u\|^2_{\dot{H}^{k/2}})$ factorizes as an infinite product when we express u as a Fourier series, we can define the re-normalization of $\exp(-\|u\|^2_{\dot{H}^{k/2}})du$ as the Gaussian measure induced by the random Fourier series

$$\varphi_{k/2}(x,\omega)=\sum_{n\neq 0}\frac{\varphi_n(\omega)}{|n|^{k/2}}e^{\mathrm{i}nx} \tag{1.3}$$

(one may ignore the zero Fourier mode since the mean of u is conserved by the flow of (1.1)). In (1.3), $(\varphi_n(\omega))_{n\neq 0}$ is a sequence of standard complex Gaussian variables defined on a probability space $(\Omega, \mathcal{A}, p)$ such that $\varphi_n=\overline{\varphi_{-n}}$ (since the solutions of (1.1) should be real valued) and $(\varphi_n(\omega))_{n>0}$ are independent. Let us denote by $\mu_{k/2}$ the measure induced by (1.3). One may easily check that $\mu_{k/2}(H^s)=1$ for every $s<(k-1)/2$ while $\mu_{k/2}(H^{(k-1)/2})=0$.

In view of the previous discussion, one may consider $\exp(-R_{k/2}(u))d\mu_{k/2}$ as a candidate of invariant measure for (1.1). There are two obstructions to do that, the first one already appears in previous works on the NLS equation (see [2, 8]) and the KdV equation (see [15]), while the second one is specific to the Benjamin-Ono equation. The first obstruction is that $\exp(-R_{k/2}(u))$ is not integrable with respect to $d\mu_{k/2}(u)$. This problem may be resolved by restricting to invariant sets, which means to replace $\exp(-R_{k/2}(u))$ by

$$\prod_{j=0}^{k-1}\chi_R(E_{j/2}(u))e^{-R_{k/2}(u)}\ , \tag{1.4}$$

where χ_R is a cut-off function defined as $\chi_R(x)=\chi(x/R)$ with $\chi:\mathbb{R}\to\mathbb{R}$ a continuous, compactly supported function such that $\chi(x)=1$ for every $|x|<1$. In the context of KdV or NLS, the function defined in (1.4) is integrable with respect to the corresponding Gaussian measure. Moreover if one takes the reunion over $R>0$ of the supports of the functions (1.4), then one obtains a set containing the support of $\mu_{k/2}$. However, in the context of the Benjamin-Ono equation, the restriction to invariant sets does not work as in (1.4) because for every R the following occurs: $\chi_R(E_{(k-1)/2}(u))=0$ almost surely on the support of $\mu_{k/2}$. One of the main points of this paper is to resolve this difficulty. This will be possible since one controls the way that $E_{(k-1)/2}(u)$ diverges on the support of $\mu_{k/2}$. More precisely, for $N\geq 1$ and $k\geq 2$, we introduce the function

$$F_{k/2,N,R}(u)=\Big(\prod_{j=0}^{k-2}\chi_R(E_{j/2}(\pi_N u))\Big)\chi_R(E_{(k-1)/2}(\pi_N u)-\alpha_N)e^{-R_{k/2}(\pi_N u)} \tag{1.5}$$

where $\alpha_N=\sum_{n=1}^{N}\frac{1}{n}$ and π_N is the Dirichlet projector on Fourier modes n such that $|n|\leq N$. Here is our first result.

THEOREM 1.1. – *For every $k\in\mathbb{N}$ with $k\geq 2$, there exists a $\mu_{k/2}$ measurable function $F_{k/2,R}(u)$ such that $F_{k/2,N,R}(u)$ converges to $F_{k/2,R}(u)$ in $L^q(d\mu_{k/2})$ for every $1\leq q<\infty$. In particular $F_{k/2,R}(u)\in L^q(d\mu_{k/2})$. Moreover, if we set $d\rho_{k/2,R}\equiv F_{k/2,R}(u)d\mu_{k/2}$, we have*

$$\bigcup_{R>0}\mathrm{supp}(\rho_{k/2,R})=\mathrm{supp}(\mu_{k/2}).$$

The above result for $k=1$ was obtained by the first author in [14]. Many of the probabilistic techniques involved in the proof of Theorem 1.1 are inspired by [3]. We also refer to [4] where in the context of the $2d$ NLS the authors use the Wick ordered L^2-cutoff, i.e., a truncation of the L^2-norm that depends on the parameter N.

We conjecture that the measures $\rho_{k/2,R}$, $k = 2, 3, \cdots$ constructed in Theorem 1.1 are invariant by the flow of the Benjamin-Ono equation established by Molinet [11], at least for even values of k. In the sequel, for shortness, we denote $\rho_{k/2,R}$ by $\rho_{k/2}$.

1.2. A property on the support of the measures

Let us now give our argument in support of the above-stated conjecture. For $N \geq 1$, we introduce the truncated Benjamin-Ono equation:

$$\partial_t u + H\partial_x^2 u + \pi_N\big((\pi_N u)\partial_x(\pi_N u)\big) = 0. \tag{1.6}$$

As in [6], one can define a global solution of (1.6) for every initial data $u(0) \in L^2(S^1)$. Indeed, one obtains that $(1-\pi_N)u(t)$ is given by the free Benjamin-Ono evolution with data $(1-\pi_N)u(0)$, while $\pi_N u(t)$ evolves under an N-dimensional ODE. This ODE has a well-defined global dynamics since the L^2 norm is preserved.

The main problem that appears when one tries to prove the invariance of $\rho_{k/2}$ is that even if $E_{k/2}$ are invariants for the Benjamin-Ono equation they are not invariant under (1.6). The invariance, however, holds in a suitable asymptotic sense as we explain below. Let us introduce the real-valued function $G_{k/2,N}$, measuring the lack of conservation of $E_{k/2}$ under the truncated flow (1.6), via the following relation

$$\frac{d}{dt}E_{k/2}(\pi_N u(t)) = G_{k/2,N}(\pi_N u(t)), \tag{1.7}$$

where $u(t)$ solves (1.6).

Denote by Φ_N the flow of (1.6) and set $d\rho_N(u) \equiv F_{k/2,N,R}(u)d\mu_{k/2}(u)$ so that by Theorem 1.1, ρ_N converges in a strong sense to $\rho_{k/2}$ (the densities converge in any $L^p(d\mu_{k/2})$, $p < \infty$). By using the Liouville theorem, one shows that for every $\mu_{k/2}$ measurable set A,

$$\rho_N(\Phi_N(t)(A)) = \int_A e^{-\int_0^t G_{k/2,N}(\pi_N\Phi_N(\tau)(u(0))d\tau} d\rho_N(u(0)) + o(1)\,.$$

Hence, a main step towards a proof of the invariance of $\rho_{k/2}$ is to show that

$$\int_0^t G_{k/2,N}(\pi_N u(\tau))d\tau \tag{1.8}$$

converges to zero, where $u(\tau)$ is a solution of (1.6), with $u(0)$ on the support of $\mu_{k/2}$. Such a property is relatively easy to be established if $u(0)$ has slightly more regularity than the typical Sobolev regularity on the support of $\mu_{k/2}$. At the present moment, we are not able to prove such a property on the support of $\mu_{k/2}$. We shall, however, prove it if we make a first approximation which consists of replacing $u(\tau)$ by $u(0)$ in (1.8). Here is the precise statement.

THEOREM 1.2. – *For every $k \geq 6$ an even integer, we have*

$$\lim_{N\to\infty} \|G_{k/2,N}(\pi_N u)\|_{L^q(d\mu_{k/2})} = 0, \ \forall q \in [1,\infty),$$

where $G_{k/2,N}$ is defined by (1.7).

Let us remark that the lack of invariance of conservation laws for the corresponding truncated flows is a problem that appears also in other contexts. We refer in particular to the papers [12] and [15], where this difficulty is resolved in the cases of the DNLS and KdV equations respectively.