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A HOMOLOGICAL STUDY OF GREEN POLYNOMIALS*

BY SYU KATO

ABSTRACT. – We interpret the orthogonality relation of Kostka polynomials arising from complex reflection groups ([51, 52] and [35]) in terms of homological algebra. This leads us to the notion of Kostka system, which can be seen as a categorical counterpart of Kostka polynomials. Then, we show that every generalized Springer correspondence ([34]) in a good characteristic gives rise to a Kostka system. This enables us to see the top-term generation property of the (twisted) homology of generalized Springer fibers, and the transition formula of Kostka polynomials between two generalized Springer correspondences of type BC. The latter provides an inductive algorithm to compute Kostka polynomials by upgrading [16] §3 to its graded version. In the appendices, we present purely algebraic proofs that Kostka systems exist for type A and asymptotic type BC cases, and therefore one can skip geometric sections §3–5 to see the key ideas and basic examples/techniques.

RÉSUMÉ. – La relation d'orthogonalité des polynômes de Kostka émanant des groupes de réflexions complexes ([51, 52] et [35]) est interprétée en termes d'algèbre homologique. Ceci nous conduit à la notion de système Kostka, qui peut être considérée comme une contrepartie catégorique des polynômes de Kostka. Puis, nous démontrons que chaque correspondance de Springer généralisée ([34]) dans une bonne caractéristique engendre un système de Kostka. Nous pouvons ainsi observer la propriété de génération du premier terme de l'homologie (tordue) des fibres de Springer généralisées, ainsi que la formule de transition de polynômes de Kostka entre deux correspondances de Springer généralisées de type BC. Cette dernière fournit un algorithme inductif de calcul des polynômes de Kostka par la mise à niveau de [16] §3 à sa version graduée. Dans les annexes, nous apportons les preuves algébriques que les systèmes de Kostka existent pour les cas de type A et de type BC asymptotique. Aussi, il est possible d'omettre de lire les sections géométriques 3 à 5 et pour entrevoir les idées-clés et parcourir des exemples/techniques de base.

* The word "green" means 'midori' in Japanese.

Introduction

Green polynomials attached to a connected reductive group over a finite field is a family of polynomials indexed by two conjugacy classes of their (little) Weyl groups[(1)], depending on a variable t that corresponds to (some twist of) the cardinality of the base field. Introduced by Green [28] for $GL(n, \mathbb{F}_q)$ and by Deligne-Lusztig [21] in general, they play a central role in the representation theory of finite groups of Lie types, affine Hecke algebras, p-adic groups, and so on. Equivalent to Green polynomials are Kostka polynomials ([28, 38]), which are t-analogues of Kostka numbers (Kostka-Foulkes polynomials) in the case of $GL(n)$ (cf. [43] Chapter III). Hence, they appear almost everywhere in representation theory attached to root data.

Despite their natural appearance, not much is known about Kostka polynomials except for type A. One major reason seems to be the fact that the set of Kostka polynomials admits integral parameters, which actually yield different collections of polynomials even if they arise from character sheaves of Chevalley groups over finite fields ([34, 35, 38]). In such representation theoretic situation, Lusztig [34] introduced the notion of symbols, which govern the combinatorial data to determine Kostka polynomials by means of their *orthogonality relation* ([51, 35]). It is generalized by Malle [44] and Shoji [52, 53] to include the case of complex reflection groups, in which the orthogonality relation is employed as their definition.

In [5], Arthur initiated now so-called elliptic representation theory, that is the "cuspidal quotient" of (usual) representation theory. Green polynomials, in the guise of characters of discrete series representations, also appear in the context of elliptic representation theory ([50, 18]). In particular, the study of formal degrees of affine Hecke algebras/p-adic groups ([47, 48, 16]) revealed the transition pattern of Kostka polynomials evaluated at $t = 1$. This supplies connections among representation theories of infinitely many p-adic groups (of different types).

The goal of the present paper is two-fold: One is to afford an algebraic framework of the study of Green/Kostka polynomials of complex reflection groups. The other is to exhibit how the classical results on Kostka polynomials of Weyl groups and the above transition pattern unveil their finer versions in our framework. From these, we expect that our framework is suited to study global structures of families of (the sets of) Kostka polynomials, and to study their connections with elliptic/usual representation theory of reductive groups or "spetses" ([13]).

For more detailed explanation, we need notations: Let W be a complex reflection group, and let $\mathsf{Irr}\, W$ denote the set of isomorphism classes of irreducible W-modules. For each $\chi \in \mathsf{Irr}\, W$, we denote by $\chi^{\vee}$ its dual representation. Let $\mathfrak{h}$ be a reflection representation of W. Form a graded algebra $A_W := \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^*]$ with $\deg w = 0$ ($w \in W$) and $\deg x = 2$ ($x \in \mathfrak{h}$). Let A_W-gmod be the category of finitely generated $\mathbb{Z}$-graded A_W-modules. For $E, F \in A_W$-gmod, we define

$$\langle E, F \rangle_{\mathsf{gEP}} := \sum_{i \geq 0} (-1)^i \mathsf{gdim}\, \mathrm{ext}^i_{A_W}(E, F) \in \mathbb{Z}(\!(t^{1/2})\!),$$

(1) The subgroup of the Weyl group that preserves the cuspidal datum (cf. §3). In case the nilpotent orbit in the cuspidal datum is $\{0\}$, then it coincides with the whole Weyl group.

where ext means the graded extension (defined so that forgetting graded vector space structure yields the usual extension; see §1.1), and gdim means the graded dimension (which sends a $\mathbb{Z}$-graded vector space $V = \oplus_{j \gg -\infty} V_j$ to $\sum_j t^{j/2} \dim V_j$). For each $\chi \in \mathsf{Irr}\, W$, we denote by L_χ the irreducible graded A_W-module sitting at degree 0 that is isomorphic to χ as a W-module.

Definition A ($\doteq$ Definition 2.13). – Let $<$ be a total pre-order on $\mathsf{Irr}\, W$. Then, a Kostka system $\{K_\chi^\pm\}_\chi \subset A_W\text{-}\mathsf{gmod}$ is a collection such that

1. Each $K_\chi^\pm$ is an indecomposable A_W-module with simple head L_χ;
2. For each $\chi, \eta \in \mathsf{Irr}\, W$, we have equalities

$$[K_\chi^+] = [L_\chi] + \sum_{\eta > \chi} K_{\chi,\eta}^+ [L_\eta] \quad \text{with} \quad K_{\chi,\eta}^+ \in t\mathbb{N}[t] \text{ and}$$
$$[K_{\chi^\vee}^-] = [L_{\chi^\vee}] + \sum_{\eta > \chi} K_{\chi,\eta}^- [L_{\eta^\vee}] \quad \text{with} \quad K_{\chi,\eta}^- \in t\mathbb{N}[t]$$

   in the Grothendieck group of $A_W\text{-}\mathsf{gmod}$;
3. We have $\langle K_\chi^+, (K_\eta^-)^* \rangle_{\mathsf{gEP}} = 0$ for $\chi \not\sim \eta^\vee$, where $(K_\eta^-)^*$ is the graded dual of K_η^-.

If W is a real reflection group, then we have $K_\chi^+ = K_\chi^-$ by (the genuine) definition, and we denote them by K_χ.

This definition is slightly weaker than the one presented in the main body of the paper (for simplicity). For Weyl groups, the classical preorders on $\mathsf{Irr}\, W$ reflect the geometry of nilpotent cones and the Springer correspondences.

Theorem B (= Theorem 2.17). – *For a Kostka system $\{K_\chi^\pm\}_\chi$, its graded character multiplicities $K_{\chi,\eta}^\pm$ satisfy the orthogonality relation of Kostka polynomials in the sense of* [51, 35, 52]. *In particular, a Kostka system is an enhancement of Kostka polynomials.*

There are a number of (conjectural) cases where Kostka polynomials of complex reflection groups satisfy the positivity of their coefficients ([44, 52, 53]). Theorem B supplies a possible framework in which such Kostka polynomials might obtain mathematical reality.

This possibility is supported by the following results that most of the Kostka polynomials in representation theory of reductive groups give rise to Kostka systems by giving graded categorifications of many of their properties:

Theorem C (= part of Theorem 3.5 and Corollary 3.9). – *Every set of Kostka polynomials arising from character sheaves of a connected reductive group over a finite field $\mathbb{F}$ admits a realization as a Kostka system whenever* $\mathsf{char}\, \mathbb{F}$ *is good. In addition, such Kostka systems are semi-orthogonal in the sense*

$$\mathrm{ext}^\bullet_{A_W}(K_\chi, K_\eta) = \{0\} \quad \text{if} \quad \chi < \eta. \tag{0.1}$$

Remark D. – Note that for a Weyl group of type A_n, the set of Kostka polynomials is unique up to tensoring sgn, while for a Weyl group of type BC_n, we have at least $4(n-1)$ different sets of Kostka polynomials.