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p-ADIC ANALYTIC TWISTS AND STRONG SUBCONVEXITY

BY Valentin BLOMER AND Djordje MILIĆEVIĆ

Abstract. – Let f be a fixed cuspidal (holomorphic or Maaß) newform. We prove a Weyl-exponent subconvexity bound $L(f \otimes \chi, 1/2 + it) \ll_{p,t} q^{1/3+\varepsilon}$ for the twisted L-function of f with a Dirichlet character χ of prime power conductor $q = p^n$ (with an explicit polynomial dependence on p and t). We obtain our result by exhibiting strong cancellation between the Hecke eigenvalues of f and the values of χ, which act as twists by exponentials with a p-adically analytic phase. Among the tools, we develop a general result on p-adic approximation by rationals (a p-adic counterpart to Farey dissection) and a p-adic version of van der Corput's method for exponential sums.

Résumé. – Soit f une forme primitive nouvelle (holomorphe ou de Maass). Soient p un nombre premier, $n \geqslant 1$ un entier, et t un nombre réel. Nous démontrons une borne sous-convexe de type Weyl pour la fonction L de f, tordue par un caractère de Dirichlet χ de conducteur $q = p^n$. Plus précisément, on démontre $L(f \otimes \chi, 1/2 + it) \ll_{p,t} q^{1/3+\varepsilon}$, avec une dépendance polynomiale et explicite en p et t. La preuve repose sur la compensation entre les valeurs propres de Hecke de f et les valeurs de χ, dont l'oscillation est gouvernée par une phase p-adique analytique. Au cours de la démonstration, on développe quelques outils p-adiques, analogues de méthodes classiques ou archimédiennes, telles que la dissection de Farey et la méthode de van der Corput pour les sommes d'exponentielles.

1. Introduction

1.1. Orthogonality of arithmetic functions

It is a central question in number theory to understand the asymptotic distribution of arithmetic functions such as the Möbius function, Dirichlet characters of large conductor, or Hecke eigenvalues of automorphic forms. It is expected that they display a certain degree of randomness, and one also expects a certain degree of (asymptotic) orthogonality between

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classes of sufficiently independent arithmetic functions. On average, this can often be proved in a strong quantitative sense by large sieve inequalities.

In this paper we are interested in convolutions of Hecke eigenvalues $a(m)$ of automorphic forms for the group $\mathrm{SL}_2(\mathbb{Z})$ and arithmetic functions g that are periodic modulo a large prime power $q = p^n$. Such arithmetic weight functions (possibly with a general defining modulus q) have been studied recently in various contexts for instance in [5, 14, 13]. We develop methods to exhibit cancellation in sums of the type

$$\sum_{\substack{m\leqslant M\\(m,p)=1}} a(m)g(m), \qquad g:(\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathbb{C}, \tag{1.1}$$

where g is a "twisting" function satisfying certain natural conditions in small p-adic neighborhoods (which we discuss in this introduction), and M is comparatively small in terms of q. As a prototypical example we work out in full detail the case where $g = \chi$ is a primitive Dirichlet character of large conductor p^n. Our main result is as follows.

THEOREM 1. – *Let p be an odd prime. Let f be a holomorphic or Maaß Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$ with Hecke eigenvalues $a(m)$, let χ be a primitive character modulo $q = p^n$, let W be a smooth weight function with support in $[1,2]$ satisfying $W^{(j)} \ll Z^j$ for some $Z \geqslant 1$, and let $M \geqslant 1$. Then*

$$L := \sum_m a(m)\chi(m)W\left(\frac{m}{M}\right) \ll_{f,\varepsilon} Z^{5/2}p^{7/6}\cdot M^{1/2}q^{1/3+\varepsilon} \tag{1.2}$$

for any $\varepsilon > 0$.

We outline our approach in the proofs of Theorems 1 and 2 (see below) in Section 1.3. Theorem 1 is properly seen as the p-adic analogue of cancellation in Dirichlet polynomials of the type

$$\sum_{m\leqslant M} a(m)m^{it} \tag{1.3}$$

for large $t \in \mathbb{R}$, which have occupied an important place in number theory and, in particular, in connection with subconvexity of automorphic L-functions. Our twists in (1.2) are p-adically analytic (see (2.2)), and Theorem 1 is a true analogue of bounds on (1.3) in the sense that we study a twist that is highly ramified at one fixed place of $\mathbb{Q}$. It should come as no surprise that Theorem 1, which establishes strong asymptotic orthogonality between Hecke eigenvalues and p-adic twists, will both be of independent interest and have applications to subconvexity, which we describe in Section 1.2.

In this section, we proceed to discuss four aspects of Theorem 1: the crucial ranges and dependence on various parameters, its relationship to the automorphic nature of f, its place within the more general framework of p-adically analytic twists of the form (1.1), and the related theory of algebraic twists.

We begin by commenting on the ranges of various parameters in Theorem 1. In this paper, q is the basic parameter, and we think of p and Z as being relatively small. We do emphasize right away, however, that all results, including Theorems 1 and 2, are completely uniform across all primes p and all prime powers $q = p^n$ (as well as across all values of Z in Theorem 1). Thus, while particularly strong results are obtained in the so-called "depth aspect", taking p fixed and having n tend to infinity, we obtain at the same time new results already for n moderately large, say with such n fixed and p tending to infinity.

The Rankin–Selberg bound (2.9) implies that $L \ll_f M$, so that (1.2) yields a non-trivial result in the range

$$M \geqslant Z^5 p^{7/3} \cdot q^{2/3+\delta} \tag{1.4}$$

for $\delta > 0$. On the other hand, if M is substantially larger than Zq, one can apply the functional equation of $L(f \otimes \chi, s)$ to reduce the length of the sum to about Z^2q^2/M. We present the details of this well-known argument at the beginning of Section 5 below, and conclude from this discussion that the real value of Theorem 1 lies in the range $q^{2/3+\delta} \ll_{Z,p} M \ll_{Z,p} q^{4/3-\delta}$.

While we did not try to optimize the exponent of the parameter Z in Theorem 1, the (explicit) polynomial dependence on Z gives us the flexibility to have slightly oscillating weight functions or weight functions with sharp cut-offs. In particular, in the situation of Theorem 1 we obtain

$$\sum_{m \leqslant M} a(m)\chi(m) \ll_{f,\varepsilon} p^{1/3} \cdot M^{6/7} q^{2/21+\varepsilon} \tag{1.5}$$

which beats the trivial bound if $M \geqslant p^{7/3}q^{2/3+\delta}$, cf. (1.4).

We put considerable care into the exponent of p in Theorem 1, although we do not claim that it is the best obtainable from our method. Finally, the implied constant in (1.2) depends polynomially on the archimedean parameter of f (weight or Laplacian eigenvalue). This can be seen by using the uniform bounds for Bessel functions in [18, Appendix]. Also, the case $p = 2$ can be dealt with in the same fashion at only the cost of some rather cumbersome notation; see [29] for a prototype where small primes are treated uniformly.

It should be noted that Theorems 1 and 2 (stated in the next subsection) are completely independent of bounds towards the Ramanujan conjecture. In fact, the only "automorphic information" needed are an approximate functional equation, the Voronoi summation formula, and a Rankin-Selberg-type mean value bound for Hecke eigenvalues.

We also remark that the natural but easier continuous spectrum analogues of both Theorems 1 and 2, involving the Eisenstein series and their Fourier coefficients $d_t(m) := \sum_{ab=m}(a/b)^{it}$, are known. By Mellin inversion, we have

$$\sum_m d_t(m)\chi(m)W\left(\frac{m}{M}\right) = \frac{1}{2\pi i}\int_{(1/2)} L(s+it,\chi)L(s-it,\chi)\widehat{W}(s)M^s ds.$$

According to [29], one has (the sub-Weyl) subconvexity bound $L(1/2+it,\chi) \ll A_{p,t} \cdot q^{1/6-\delta}$ for a primitive character χ of conductor $q = p^n$ with an explicit $A_{p,t} > 0$ and absolute $\delta > 0$. From this, one obtains the desired cancellation between $d_t(m)$ and characters of conductor $q = p^n$ in the situation of Theorem 1 with the even stronger bound $\ll B_{p,t,Z} \cdot q^{1/3-2\delta}M^{1/2}$ with an explicit $B_{p,t,Z} > 0$.

Our method can be adapted to treat other sums of the form (1.1), provided that the local behavior of g in small p-adic neighborhoods meets suitable conditions that we now discuss. In particular, we need to be able to control the terms that take place of the first and second derivative (which, for p-adic analytic functions, can be read off from their p-adic power series expansion) of a specific phase resulting from g; see also the discussion at the end of Section 4.