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Christian BONATTI & Kamlesh PARWANI & Rafael POTRIE

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

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ANOMALOUS PARTIALLY HYPERBOLIC DIFFEOMORPHISMS I: DYNAMICALLY COHERENT EXAMPLES

BY Christian BONATTI, Kamlesh PARWANI AND Rafael POTRIE

Abstract. – We build an example of a non-transitive, dynamically coherent partially hyperbolic diffeomorphism f on a closed 3-manifold with exponential growth in its fundamental group such that f^n is not isotopic to the identity for all $n \neq 0$. This example contradicts a conjecture in [13]. The main idea is to consider a well-understood time-t map of a non-transitive Anosov flow and then carefully compose with a Dehn twist.

Résumé. – Sur une 3-variété fermée dont le groupe fondamental est à croissance exponentielle, nous construisons un exemple de difféomorphisme f, partiellement hyperbolique, dynamiquement cohérent, non transitif, et dont aucune puissance f^n, $n \neq 0$, n'est isotope à l'identité. Cet exemple infirme une conjecture de [13]. L'exemple est obtenu en composant avec soin le temps t d'un flot d'Anosov non transitif bien choisi avec un twist de Dehn.

1. Introduction

In recent years, partially hyperbolic diffeomorphisms have been the focus of considerable study. Informally, partially hyperbolic diffeomorphisms are generalization of hyperbolic maps. The simplest partially hyperbolic diffeomorphisms admit an invariant splitting into three bundles: one of which is uniformly contracted by the derivative, another which is uniformly expanded, and a center direction whose behavior is intermediate. A more formal definition will soon follow. In this paper, we shall restrict to the case of these diffeomorphisms on closed 3-dimensional manifolds.

The study of partially hyperbolic diffeomorphisms has followed two main directions. One consists of studying conditions under which a volume preserving partially hyperbolic diffeomorphism is stably ergodic. This is not the focus of this article. See [11, 14] for recent surveys on this subject.

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The second direction, initiated in [3, 4], has as a long term goal of classifying these partially hyperbolic systems, at least topologically. Even in dimension 3, this goal seems quite ambitious but some partial progress has been made, which we briefly review below. This paper is intended to further this classification effort by providing new examples of partially hyperbolic diffeomorphisms. In a forthcoming paper ([2]) we shall provide new transitive examples (and even stably ergodic); their construction uses some of the ideas of this paper as well as some new ones. Another viewpoint might be that these new examples throw a monkey wrench into the classification program. In light of these new partially hyperbolic diffeomorphisms, is there any hope to achieve any reasonable sense of a classification?

1.1. Preliminaries

Before diving into a detailed exposition of these examples, we provide the necessary definitions and background. Let M be a closed 3-manifold, we say that a diffeomorphism $f : M \to M$ is *partially hyperbolic* if the tangent bundle splits into three one-dimensional (1) Df-invariant continuous bundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$ such that there exists $\ell > 0$ such that for every $x \in M$:

$$\|Df^\ell|_{E^{ss}(x)}\| < \min\{1, \|Df^\ell|_{E^c(x)}\|\} \leq \max\{1, \|Df^\ell|_{E^c(x)}\} < \|Df^\ell|_{E^{uu}(x)}\|.$$

Sometimes, the more restrictive notion of *absolute partial hyperbolicity* is used. This means that f is partially hyperbolic and there exists $\lambda < 1 < \mu$ such that:

$$\|Df^\ell|_{E^{ss}(x)}\| < \lambda < \|Df^\ell|_{E^c(x)}\| < \mu < \|Df^\ell|_{E^{uu}(x)}\|.$$

For the classification of such systems, one of the main obstacles is understanding the existence of invariant foliations tangent to the center direction E^c. In general, the bundles appearing in the invariant splitting are not regular enough to guaranty unique integrability. In the case of the strong stable E^{ss} and strong unstable E^{uu} bundles, dynamical arguments insure the existence of unique foliations tangent to the strong stable and unstable bundle (see for example [8]). However, the other distributions need not be integrable.

The diffeomorphism f is *dynamically coherent* if there are 2-dimensional f-invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to the distributions $E^{ss} \oplus E^c$ and $E^c \oplus E^{uu}$, respectively. These foliations, when they exist, intersect along a 1-dimensional foliation $\mathcal{W}^c$ tangent to E^c. The diffeomorphism f is *robustly dynamically coherent* if there exists a C^1-neighborhood of f comprised only of dynamically coherent partially hyperbolic diffeomorphisms. There is an example of non dynamically coherent partially hyperbolic diffeomorphisms (2) on $\mathbb{T}^3$ (see [12]). This example is not transitive and it is not known whether every transitive partially hyperbolic diffeomorphisms on a compact 3 manifold is dynamically coherent. See [7] and references therein for the known results on dynamical coherence of partially hyperbolic diffeomorphisms in dimension 3.

Two dynamically coherent partially hyperbolic diffeomorphisms $f : M \to M$ and $g : N \to N$ are *leaf conjugate* if there is a homeomorphism $h : M \to N$ so that h maps

(1) One of the advantages of working with one-dimensional bundles is that the norm of Df along such bundles controls the contraction/expansion of every vector in the bundle. Compare with definitions of partial hyperbolicity when the bundles are not one-dimensional [14].

(2) We should remark that this example is not absolutely partially hyperbolic. Moreover, it is isotopic to one of the known models of partially hyperbolic diffeomorphisms.

the center foliation of f on the center foliation of g and for any $x \in M$ the points $h(f(x))$ and $g(h(x))$ belong to the same center leaf of g.

Up to now the only known example of dynamically coherent partially hyperbolic diffeomorphisms were, up to finite lift and finite iterates, leaf conjugate to one of the following models:

1. linear Anosov automorphism of $\mathbb{T}^3$;
2. skew products over a linear Anosov map of the torus $\mathbb{T}^2$;
3. time one map of an Anosov flow.

It has been conjectured, first in the transitive case (informally by Pujals in a talk and then written in [3]), and later in the dynamically coherent case (in many talks and minicourses [13]) that every partially hyperbolic diffeomorphism should be, up to finite cover and iterate, leaf conjugate to one of these three models. Positive results have been obtained in [3, 4, 7] and some families of 3-manifolds are now known only to admit partially hyperbolic diffeomorphisms that are leaf conjugate to the models.

1.2. Statements of results

The aim of this paper is to provide a counter example to the conjecture stated above. Our examples are not isotopic to any of the models.

In order to present the ideas in the simplest way, we have chosen to detail the construction of a specific example on a (possibly the simplest) manifold admitting a non-transitive Anosov flow transverse to a non-homologically trivial incompressible two-torus; the interested reader should consult [5] for more on 3-manifolds admitting such non-transitive Anosov flows. Our arguments go through directly in some other manifolds, but for treating the general case of 3-manifolds admitting non-transitive Anosov flows further work must be done.

THEOREM 1.1. – *There is a closed orientable* 3*-manifold M endowed with a non-transitive Anosov flow X and a diffeomorphism $f\colon M \to M$ such that:*

- *f is absolutely partially hyperbolic;*
- *f is robustly dynamically coherent;*
- *the restriction of f to its chain recurrent set coincides with the time-one map of the Anosov flow X, and*
- *for any $n \neq 0$, f^n is not isotopic to the identity.*

The manifold M on which our example is constructed also admits a transitive Anosov flow (see [1]).

As a corollary of our main theorem, we show that f is a counter example to the conjecture stated above in the non-transitive case (see [13, 7]):

COROLLARY 1.2. – *Let f be the diffeomorphism announced in Theorem 1.1. Then for all n the diffeomorphism f^n does not admit a finite lift that is leaf conjugate to any of the following:*

- *linear Anosov diffeomorphisms on $\mathbb{T}^3$;*
- *partially hyperbolic skew product with circle fiber over an Anosov diffeomorphism on the torus $\mathbb{T}^2$;*
- *the time-one map of an Anosov flow.*