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Javier CARVAJAL-ROJAS & Karl SCHWEDE & Kevin TUCKER

*Fundamental groups of  $F$ -regular singularities via  $F$ -signature*

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# FUNDAMENTAL GROUPS OF $F$ -REGULAR SINGULARITIES VIA $F$ -SIGNATURE

BY JAVIER CARVAJAL-ROJAS, KARL SCHWEDE  
AND KEVIN TUCKER

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**ABSTRACT.** – We prove that the local étale fundamental group of a strongly  $F$ -regular singularity is finite. These results are analogous to results of Xu and Greb-Kebekus-Peternell for KLT singularities in characteristic 0. Our result is effective, we show that the reciprocal of the  $F$ -signature of the singularity gives a bound on the size of this fundamental group. To prove these results we develop new transformation rules for the  $F$ -signature under finite étale-in-codimension-one extensions. We also obtain purity of the branch locus over rings with mild singularities (particularly if the  $F$ -signature is  $> 1/2$ ).

**RÉSUMÉ.** – Nous montrons que le groupe fondamental local étale d’une singularité  $F$ -régulière est fini. Ce théorème représente l’analogie en caractéristique  $p$  des résultats obtenus par Xu et Greb-Kebekus-Peternell pour les singularités KLT. Nous montrons que le cardinal du groupe fondamental est majoré par l’inverse de la  $F$ -signature de la singularité. En particulier, notre résultat principal est effectif. Pour cela, nous établissons des nouvelles formules de transformation de la  $F$ -signature par rapport aux extensions étale en codimension un. Nous obtenons également un nouveau critère de pureté du lieu de branchement sur les anneaux à singularités faibles. Ceci s’applique en particulier aux anneaux dont la  $F$ -signature est supérieure à  $1/2$ .

## 1. Introduction

In [21, Question 26] J. Kollár asked whether if  $(0 \in X)$  is the germ of a KLT singularity, then  $\pi_1(X \setminus \{0\})$  is finite. In [41] C. Xu showed that this holds for the étale local fundamental group, in other words, for the profinite completion of  $\pi_1(X \setminus \{0\})$ . Building on this result, [11] proved the finiteness of the étale fundamental groups of the regular locus of KLT singularities (see also [35]). Over the past few decades, we have learned that KLT singularities are closely related to strongly  $F$ -regular singularities in characteristic  $p > 0$ , see [16, 15]. Hence it is natural to ask whether their local étale fundamental groups are also finite. We show that

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this is indeed the case. In fact, we find an upper bound for the size of the fundamental group in terms of a well studied invariant for measuring singularities in characteristic  $p > 0$ , the  $F$ -signature  $s(R)$ .

**THEOREM A** (Theorem 5.1). – *Let  $(R, \mathfrak{m}, k)$  be a normal  $F$ -finite and strongly  $F$ -regular strictly Henselian<sup>(1)</sup> local domain of prime characteristic  $p > 0$ , with dimension  $d \geq 2$ . Then the étale fundamental group of the punctured spectrum of  $R$ , i.e.,  $\pi_1 := \pi_1^{\text{ét}}(\text{Spec}^\circ(R))$ , is finite. Furthermore, the order of  $\pi_1$  is at most  $1/s(R)$  and is prime to  $p$ . The same also holds for  $\pi_1^{\text{ét}}(\text{Spec}(R) \setminus Z)$  where  $Z \subseteq \text{Spec } R$  has codimension  $\geq 2$ .*

Observe that unlike the characteristic zero situation, our characteristic  $p > 0$  result is effective. We give an explicit bound on the size of the  $\pi_1$ . It is also worth noting that we are working with the étale fundamental group, not the *tame* fundamental group. Indeed, for  $R$  strongly  $F$ -regular, any finite Galois étale in codimension 1 local extension  $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$  must be tame everywhere. This was already implicitly observed in [31] but we make it precise here. Indeed, we note that  $p$  cannot divide  $[K(S) : K(R)]$  if the residue fields are equal (Corollary 2.11).

The technical tool where  $F$ -regularity is used in our proof is a transformation rule for  $F$ -signature under finite étale-in-codimension-1-morphisms. The  $F$ -signature was introduced implicitly in [32] and explicitly in [18]. Roughly speaking, it measures how many different ways  $R \hookrightarrow F_*^e R$  splits as  $e$  goes to infinity. Explicitly, if  $R$  has perfect residue field and  $F_*^e R = R^{\oplus a_e} \oplus M$  as an  $R$ -module, where  $M$  has no free  $R$ -summands, then  $s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e \dim R}}$ . Here are three quick facts:

- The limit  $s(R)$  exists [37].
- $s(R) > 0$  if and only if  $R$  is strongly  $F$ -regular [1].
- $s(R) \leq 1$ .

Note that there have been a number of transformation rules for  $F$ -signature under finite maps in the past. However, they were generally only an inequality (that went the wrong way for our purposes), or assumed that  $S$  is flat over  $R$  (or made other assumptions about  $R$  and  $S$ ). See for instance [18, 42, 17, 37].

**THEOREM B** (Theorem 3.1). – *Let  $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$  be a module-finite local extension of  $F$ -finite  $d$ -dimensional normal local domains in characteristic  $p > 0$ , with corresponding extension of fraction fields  $K \subseteq L$ . Suppose  $R \subseteq S$  is étale in codimension 1, and that  $R$  is strongly  $F$ -regular. Then if one writes  $S = R^{\oplus f} \oplus M$  as a decomposition of  $R$ -modules so that  $M$  has no nonzero free direct summands, then  $f = [\ell : k] \geq 1$  and the following equality holds:*

$$s(S) = \frac{[L : K]}{[\ell : k]} \cdot s(R).$$

Below, before Theorem D, we discuss how to still get precise transformation rules of  $F$ -signature even when  $R \subseteq S$  is not necessarily étale in codimension 1.

By applying Theorem B in the case  $k = \ell$ , we see that  $s(S) = [L : K] \cdot s(R)$ . Since  $s(S) \leq 1$ , we immediately see that  $[L : K] \leq 1/s(R)$ . In other words, the reciprocal

<sup>(1)</sup> This just means it is Henselian with separably closed residue field.

of the  $F$ -signature  $s(R)$  gives an upper bound on the generic rank of a finite local étale in codimension 1 extension with the same residue field. Theorem A then follows. We also obtain characteristic  $p > 0$  corollaries similar to some of those in [11].

Because our bound on the size of the étale fundamental group is effective, we immediately obtain a new result on purity of the branch locus.

**THEOREM C (Corollary 3.3).** – *Suppose  $Y \rightarrow X$  is a finite dominant map of  $F$ -finite normal integral schemes. If  $s(\mathcal{O}_{X,P}) > 1/2$  for all  $P \in X$  then the branch locus of  $Y \rightarrow X$  has no irreducible components of codimension  $\geq 2$ , in other words it is a divisor.*

In [5], the notion of  $F$ -signature of pairs was introduced. In Theorem 4.4, we obtain an analogous result to Theorem B in the context of pairs. Indeed, if  $(R, \Delta)$  is a strongly  $F$ -regular pair, then this can be interpreted as follows. The reciprocal of  $s(R, \Delta)$  gives an upper bound on the generic rank of a finite local extension  $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$  such that  $\pi^* \Delta - \text{Ram} \geq 0$  (here  $\text{Ram}$  is the ramification divisor on  $\text{Spec } S$  and  $\pi : \text{Spec } S \rightarrow \text{Spec } R$  is the induced map). By taking cones, this immediately yields the following characteristic  $p > 0$  analog of the second main result of [41]. Here note that globally  $F$ -regular varieties are an analog of log-Fano varieties in characteristic zero [30].

**THEOREM D (Corollary 4.8).** – *Suppose that  $(X, \Delta)$  is a globally  $F$ -regular projective pair over an algebraically closed field of characteristic  $p > 0$ . There is a number  $n$  such that every finite separable cover  $\pi : Y \rightarrow X$  with  $\pi^* \Delta - \text{Ram} \geq 0$  has generic rank  $[K(Y) : K(X)] \leq n$ .*

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## 2. Preliminaries

**CONVENTION 2.1.** – Throughout this paper, all rings will be assumed to be Noetherian. They will all be characteristic  $p > 0$  unless otherwise stated and they will all be  $F$ -finite. All schemes will be assumed to be Noetherian and separated. If  $R$  is an integral domain, then  $K(R)$  will denote the fraction field of  $R$  (likewise with  $K(X)$  if  $X$  is an integral scheme). Given a finite separable map of normal integral schemes  $f : Y \rightarrow X$ , we use  $\text{Ram}$  to denote the ramification divisor on  $Y$ .