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Benjamin DODSON

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

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GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE FOCUSING, CUBIC SCHRÖDINGER EQUATION IN DIMENSION $d = 4$

BY BENJAMIN DODSON

ABSTRACT. – In this paper we prove global well-posedness and scattering for the focusing, cubic Schrödinger equation in four dimensions below the ground state. Previous work proved this in five dimensions and higher. To prove this, we combine the double Duhamel method with the long time Strichartz estimates.

RÉSUMÉ. – Nous prouvons l'existence globale et la diffusion des ondes pour l'équation de Schrödinger cubique focalisante en dimension quatre. Des travaux antérieurs ont montré de tels résultats en dimension supérieure ou égale à cinq. Nous utilisons ici la méthode de Duhamel double et les estimations de Strichartz en temps long.

1. Introduction

In this paper we study the nonlinear Schrödinger initial value problem

$$
\begin{aligned}
& i u_t + \Delta u = F(u) = -|u|^2 u, \\
& u(0, x) = u_0 \in \dot{H}^1(\mathbf{R}^4),
\end{aligned}
\tag{1.1}
$$

which belongs to a class of problems known as the focusing, nonlinear Schrödinger initial value problems,

$$
\begin{aligned}
& i u_t + \Delta u = F(u) = -|u|^p u, \\
& u(0, x) = u_0 \in \dot{H}^1(\mathbf{R}^d),
\end{aligned}
\tag{1.2}
$$

In general a solution to (1.2) conserves mass,

$$
M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)), \tag{1.3}
$$

and energy,

$$
E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx - \frac{1}{p+2} \int |u(t,x)|^{p+2} dx = E(u(0)). \tag{1.4}
$$

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When $p = \frac{4}{d-2}$, (1.2) is called energy-critical since a solution to (1.2) is invariant under the scaling

$$u(t,x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x), \tag{1.5}$$

and (1.5) preserves the energy (1.4) when $p = \frac{4}{d-2}$.

The global behavior of the defocusing, energy-critical problem ($F(u) = |u|^{\frac{4}{d-2}} u$) is now completely worked out for any $d \geq 3$.

THEOREM 1.1. – *The defocusing initial value problem* (1.2), $F(u) = |u|^{\frac{4}{d-2}} u$, *is globally well-posed for any* $u_0 \in \dot{H}^1(\mathbf{R}^d)$, $d \geq 3$, *and the solution scatters both forward and backward in time.*

DEFINITION 1.1 (Scattering). – *A solution* u *to* (1.2) *with* $p = \frac{4}{d-2}$ *is said to scatter forward in time if there exists* $u_+ \in \dot{H}^1$ *such that*

$$\lim_{t \nearrow +\infty} \| u(t) - e^{it\Delta} u_+ \|_{\dot{H}^1(\mathbf{R}^d)} = 0. \tag{1.6}$$

Likewise, u *is said to scatter backward in time if there exists* $u_- \in \dot{H}^1$ *such that*

$$\lim_{t \searrow -\infty} \| u(t) - e^{it\Delta} u_- \|_{\dot{H}^1(\mathbf{R}^d)} = 0. \tag{1.7}$$

Proof. – The proof of Theorem 1.1 involved contributions from numerous authors. [10] proved Theorem 1.1 for small data for both the focusing and defocusing problem. [10] also proved that (1.2) has a local solution for any initial data $u_0 \in \dot{H}^1(\mathbf{R}^d)$, where the time of existence depends on the size and profile of u_0.

For large data, the seminal result was the work of [5] (also see [4]), proving Theorem 1.1 for radial data in dimensions $d = 3, 4$, and also that for more regular u_0, this additional smoothness is preserved. See [23] for another proof of this last fact. [42] then extended Theorem 1.1 to radial data in higher dimensions. Both [5] and [42] used the induction on energy method.

For nonradial data, the first progress came when [13] extended Theorem 1.1 to general $u_0 \in \dot{H}^1(\mathbf{R}^3)$. Subsequently, [36] extended this result dimension $d = 4$, and [49] (also see [48]) extended Theorem 1.1 to dimensions $d \geq 5$. □

REMARK. – [31] and [50] have since used the long time Strichartz estimates of [14] to reprove Theorem 1.1 in dimensions three and four, respectively.

However, Theorem 1.1 does not hold for arbitrary data in the focusing case. By the virial identity (see for example [22])

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 8[\int |\nabla u(t,x)|^2 dx - \int |u(t,x)|^{\frac{2d}{d-2}} dx], \tag{1.8}$$

so if $xu_0 \in L^2(\mathbf{R}^d)$ and $E(u_0) < 0$, $\int |x|^2 |u(t,x)|^2 dx$ is a function of t that is concave down and has two real roots, $t_1 < 0 < t_2$. Then the positive definiteness of $\int |x|^2 |u(t,x)|^2 dx$ implies that the solution to (1.1) with such u_0 cannot exist outside of $[t_1, t_2]$.

There also exist global solutions to (1.1) that do not scatter.

$$W(x) = \frac{1}{(1 + \frac{|x|^2}{d(d-2)})^{\frac{d-2}{2}}} \tag{1.9}$$

lies in $\dot{H}^1(\mathbf{R}^d)$ and solves the elliptic equation

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0, \tag{1.10}$$

so $W(x,t) = W(x)$ solves (1.1) but is clearly non scattering. Therefore, as in the mass-critical problem we conjecture that scattering holds for initial data below the threshold given by (1.9).

Conjecture 1.1. – *Let $d \geq 3$ and let $u : I \times \mathbf{R}^d \to \mathbf{C}$ be a solution to* (1.2), $p = \frac{4}{d-2}$. *If*

$$\|u_0\|_{\dot{H}^1(\mathbf{R}^d)} < \|W\|_{\dot{H}^1(\mathbf{R}^d)}, \tag{1.11}$$

and

$$E(u_0) < E(W), \tag{1.12}$$

then

$$\int_I \int |u(t,x)|^{\frac{2(d+2)}{d-2}} dx dt \leq C(\|u_0\|_{\dot{H}^1}, E(u_0)) < \infty. \tag{1.13}$$

The quantity $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbf{R}\times\mathbf{R}^d)}$ is the key quantity to determining whether or not u scatters forward in time or backward in time.

Definition 1.2 (Scattering size). – *The scattering size of a solution to* (1.2) *on a time interval I is given by*

$$S_I(u) = \int_I \int_{\mathbf{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} dx dt. \tag{1.14}$$

Theorem 1.2. – *When $p = \frac{4}{d-2}$,* (1.2) *is well-posed on some open interval $I(u_0)$. Additionally, u scatters forward in time if and only if $S_{[t_1,\infty)}(u) < \infty$ for some $t_1 \in \mathbf{R}$. Likewise, u scatters backward in time if and only if $S_{(-\infty,t_1]}(u) < \infty$ for some $t_1 \in \mathbf{R}$.*

Proof. – See [10] and [11]. □

Therefore a solution may either scatter or blow-up.

Definition 1.3 (Blow up). – *A solution u to* (1.2) *blows up forward in time on I if there exists $t_1 \in I$ such that*

$$S_{[t_1,\sup(I))}(u) = \infty. \tag{1.15}$$

u blows up backward in time if there exists $t_1 \in I$ such that

$$S_{(\inf(I),t_1]}(u) = \infty. \tag{1.16}$$

[25] proved Conjecture 1.1 for radial data in dimensions $d = 3, 4, 5$. The proof uses the concentration compactness argument.

Theorem 1.3. – *Assume that $E(u_0) < E(W)$, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, $d = 3, 4, 5$, and u_0 is radial. Then* (1.2) *is globally well-posed and scatters forward and backward in time.*

Proof. – See [25]. □

[28] treated the nonradial case in dimensions $d \geq 5$.

Theorem 1.4. – *Assume that $E(u_0) < E(W)$, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, $d \geq 5$. Then* (1.2) *is globally well-posed and scatters forward and backward in time.*