DOCUMENTS MATHÉMATIQUES

RÉSIDUS ET DUALITÉ
Prénotes pour un « Séminaire Hartshorne »

Alexandre Grothendieck
Édité par Robin Hartshorne

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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DOCUMENTS MATHÉMATIQUES 21

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5.4. Modules et Pseudo-complexes fondamentaux relatifs pour un morphisme. Morphismes localement de dimension cohomologique relative finie, morphismes cohomologiquement de Cohen-Macaulay, de Gorenstein.

Définition 5.4.1. Soit f un morphisme localement de type fini $X \to Y$. On appelle Pseudocomplexe fondamental de f, ou de X relativement à f, le Pseudo-complexe $f^{!}(\underline{O}_Y)$. (Observer que sa cohomologie est souvent nulle en degrés > 0, c'est pourquoi il est parfois commode d'utiliser la notation inférieure pour les degrés). Soit n un entier tel que $\underline{H}_i(f^{!}(\underline{O}_Y)) = 0$ pour $i > n$, alors $\underline{H}_n(f^{!}(\underline{O}_Y))$ est appelé Module fondamental relatif de f, ou de X relativement à Y, en dimension n, et noté $\underline{\omega}_{n\, X/Y}$ ou $\underline{\omega}^{f}_{n}$, ou $\underline{\omega}_{X/Y}$ si aucune confusion n'est à craindre.

Bien entendu, le seul cas intéressant est celui où n est le plus petit entier ayant la propriété dite, i.e. le cas où $\underline{\omega}_{n\, X/Y}$ est $\neq 0$; lorsqu'on parlera de Module fondamental $\underline{\omega}_{X/Y}$ de X/Y sans préciser la dimension, il sera sous-entendu, sauf mention du contraire, qu'il s'agit de celui-là. C'est pour la simplicité de certaines formulations qu'il est bon cependant d'introduire la notation $\underline{\omega}_{n\, X/Y}$ pour un entier n général (ainsi, la formation de $\underline{\omega}_{n\, X/Y}$ commute avec la restriction à un ouvert, alors qu'il n'en est plus de même de la formation de $\underline{\omega}_{X/Y}$, car l'entier n "critique" peut devenir plus petit en se restreignant à un ouvert). Bien entendu, rien n'empêche de poser, pour tout n entier

$$\underline{\omega}_{n\, X/Y} = \underline{H}_n(f^{!}(\underline{O}_Y)) \quad ,$$

mais si n n'a pas la propriété précisée plus haut, il ne faut plus appeler ce Module "fondamental". Les Modules fondamentaux jouent un rôle important tant au point de vue global (par exemple dans certaines dimensions le théorème de dualité globale s'exprime à l'aide du seul Module fondamental, au lieu de $f^{!}(\underline{O}_Y)$ tout entier) que local. Bien entendu, le cas le plus intéressant sera celui où $\underline{\omega}_{n\, X/Y}$ est le seul Module de cohomologie non nul de $f^{!}(\underline{O}_Y)$,

TABLE DES MATIÈRES

PREFACE

This volume presents unpublished notes of Grothendieck dating from 1963, together with introductory remarks by Hartshorne. The circumstances of the writing of these notes and their relation to [11] are explained in the Introduction.

This document is remarkable in many ways. First of all, it is the unique substantive mathematical text of Grothendieck dating from the 1960's, written entirely by his own hand, that has not been previously published or made available in some other form. One may consider that in 1963, Grothendieck was at the summit of his mathematical creativity. In these prenotes one sees him at work developing a grand new theory with his characteristic driving force. In contrast to the correspondence with Serre [3], Grothendieck is not addressing an exceptional correspondent who can be expected to understand half-explained ideas. Rather, he is writing for ordinary mathematicians to whom he expresses himself fully. Definitions, theorems, calculations, corollaries, complements, and conjectures are linked together at breakneck speed, the whole informed by an overarching vision.

It is often said that in algebraic geometry, there is a before and an after Grothendieck, marked by the theory of schemes, the construction of geometric objects by representing functors, and the discovery and systematic use of new topologies (sites and topoi). But one usually forgets to mention a fourth pillar of this "refoundation" of algebraic geometry, namely the introduction of derived categories. Indeed a large (and perhaps essential) portion of the work in algebraic geometry and number theory since the middle of the last century has been the study of cohomological invariants of objects arising in the development of these two areas. In 1963, Grothendieck realized that in order to fully develop the generalizations he had in mind for the duality theorems of Serre in the relative case, the methods of homological algebra from the 1950's were insufficient. He then imagined the simple but revolutionary idea of "derived categories". Their construction, which he first visualized heuristically, was then developed formally, at his request, by Verdier [18], and later by many other authors. The derived categories allowed him to state and prove the theorems he wanted. Since then, they have become a fruitful tool of tremendous versatility in many branches of mathematics. The language of derived categories, which appears here for the first time, together with its associated functors, is a particular case of what Grothendieck calls in [10] the formalism of "les six opérations". Here we see their first application in the case of coherent sheaves on schemes.

We refer the reader to the Introduction for comments on the genesis of this theory.

To the extent that these prenotes gave rise to the seminar of Hartshorne [11], one can ask, beyond their historical importance, what purely mathematical interest they may have for the reader of today. Since Grothendieck had a broad vision, these notes have a certain number of definitions, developments, and conjectures that either do not appear in [11], or remain to be explored, or have been studied in later work, sometimes extensively. We provide a list of these in an appendix following this Introduction.

I would like to give warm thanks to:

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Robin Hartshorne
Berkeley, CA
January 15, 2024

INTRODUCTION TO *PRÉNOTES*

by

Robin Hartshorne

Historical background

For algebraic varieties over the complex numbers, topological methods and cohomology were already present, at least implicitly, in the work of Poincaré, Lefschetz, and later Kodaira and Spencer. Sheaves were introduced by Leray. The concept of a coherent sheaf of analytic functions on a complex manifold was arrived at independently by H. Cartan and K. Oka in the 1940's, but it was Oka who first proved the fundamental result that the structural sheaf of a complex analytic manifold is coherent. These ideas were then fully developed in the Cartan seminars of the early 1950's.

A first bit of duality appeared in the dual role played by $\ell(D)$ and $\ell(K-D)$ in the Riemann-Roch theorem, in the work of Kodaira and Spencer. Seizing on this, Serre proved a duality theorem [16] for a vector bundle V on a compact complex manifold X between $H^i(X,V)$ and $H^{n-i}(X,V^\vee\otimes\Omega^n)$ where n is the dimension of X and Ω^n is the bundle of differential forms of top dimension. His proof is analytic, using resolutions by sheaves of C^∞-differential forms and duality of Fréchet spaces. As a consequence, he gave a simple proof of the Riemann-Roch theorem for curves, using duality. This is the model for later *Serre duality* theorems in algebraic geometry.

The introduction of sheaves and cohomology into abstract algebraic geometry, that is, over fields k other than the complex numbers, is due to Serre in his seminal paper [15]. In that paper, Serre works with algebraic varieties over an algebraically closed ground field k (what we now call separated reduced schemes of finite type over k). He defined coherent sheaves of modules $\mathcal{F}$ over the structure sheaf $\mathcal{O}_X$ of a variety X. He defined cohomology using a Čech process, taking the limit over finer and finer open coverings. While the cohomology groups were defined for arbitrary sheaves of abelian groups on a topological space X, he could prove the long exact sequence of cohomology only for coherent sheaves on varieties, using the fact (which he proved) that the higher cohomology of a coherent sheaf on an affine variety is zero, the algebraic analog of *Cartan's theorem B* for Stein manifolds.

Near the end of [15], Serre proves a duality theorem comparing the cohomology of a sheaf $\mathcal{F}$ on $\mathbf{P}^n_k$ to certain Ext groups over the homogeneous coordinate ring of the ambient projective space. The Ext groups were defined in the book of Cartan and Eilenberg, and were known at that time only for modules over a ring. The statement is awkward in comparison with the later formulation using Ext of sheaves of modules, but the content is there, in the form of Serre duality for coherent sheaves on projective space.

Grothendieck's theorem in his Séminaire Bourbaki talk [5] is a significant improvement over Serre's result, though still for coherent sheaves on a nonsingular projective variety. First of all, by that time he had developed the theory of derived functors in an abstract abelian category, and thus could construct the cohomology groups of sheaves on a topological space as derived functors of the global section functor. He could also define the Ext groups of sheaves as derived functors, so that the theorem proved by Serre appeared in the much neater form : $H^i(X,\mathcal{F})$ is dual to $\mathrm{Ext}^{n-i}_X(\mathcal{F},\omega_X)$, where $\omega_X = \Omega^n_{X/k}$ is the sheaf of top differential forms. Relating the differentials on a nonsingular projective variety X to those on an ambient projective space, he was also able to state the theorem intrinsically on X, without recourse to Ext modules over the projective space. Finally, in the case of a singular variety X with suitable homological conditions on its structure sheaf (what we now call a locally Cohen-Macaulay scheme), he was able to establish the same formula, replacing the differentials Ω^n_X by an analogous sheaf ω_X defined as $\mathcal{E}xt^r_{\mathbf{P}^n}(\mathcal{O}_X,\Omega^n_{\mathbf{P}^n/k})$, where r is the codimension of X in $\mathbf{P}^n_k$. This is what we now call the *Grothendieck dualizing sheaf*. It is independent of the embedding in projective space.

Up to this point we can say that Grothendieck had greatly improved an existing theorem by his supple use of the techniques of homological algebra developed in his article Tôhoku [4]. The main point there is the systematic use of derived functors and abstract abelian categories. This theorem and its proof are essentially what is reproduced in Hartshorne's book [13, Ch. III, Sec. 7] under the heading of *Serre duality*.

It is fascinating to observe the development of these ideas, along with the nascent concept of scheme and Grothendieck's ambitious plans for a great treatise on algebraic geometry, in the now published correspondence between Serre and Grothendieck [3]. Here are a few excerpts.

18.2.55. AG to JPS. J'ai fait pour mon propre compte, en utilisant les indications que tu m'as données ici et là et les séminaires Cartan, une révision systématique (d'ailleurs pas encore terminée) de mes notions d'algèbre homologique. Ça me semble extrêmement plaisant de ficher comme ça beaucoup de choses, pas drôles quand on les prend séparément, sous le grand chapeau des foncteurs dérivés.

15.12.55. AG to JPS. [He explains his revised version of the duality in Serre's [15], expressed intrinsically on X using both cohomology and Ext as derived functors.]

22.12.55. JPS to AG. Ta formule

$$H^{n-p}(X, \mathcal{F})' = \mathrm{Ext}^p_{\mathcal{O}}(X, \mathcal{F}, \Omega^n)$$

m'excite beaucoup, car je suis bien convaincu que c'est *la* bonne façon d'énoncer le théorème de dualité à la fois dans le cas analytique et dans le cas algébrique (le plus important – pour moi).

Origin of these prenotes

In his ICM talk [6], Grothendieck already saw that for a projective variety X with arbitrary singularities, a single sheaf playing the role of $\Omega^n_{X/k}$ for a nonsingular variety would not suffice. So he proposed replacing ω_X by a *residual complex*, a complex of injective $\mathcal{O}_X$-modules K, so that for any coherent sheaf $\mathcal{F}$ on X, the cohomology $H^i(X, \mathcal{F})$ would be dual to $H^{n-i}(X, \mathcal{H}om^\bullet(\mathcal{F}, K))$. But at the same time he was unhappy with the dependence of the whole theory so far on the projective space (letter to JPS dated 12.1.56). To overcome that would require an intrinsic description of the residual complex, a development of an analogous local theory, and the proof of many functorial compatibilities (see for example Théorème conjectural 4.1.1 in these prenotes). The existing techniques of homological algebra, of abelian categories and derived functors were simply not sufficient to handle this at that time.

What he needed was a way of dealing with all the derived functors at once. Of course derived functors appear as the cohomology objects of a complex, but that complex is not unique. In some cases, it may be unique up to homotopy, but not always. It is, however, unique up to quasi-isomorphism, meaning a morphism of complexes that induces an isomorphism on all the cohomology objects. Thus Grothendieck made the bold step of imagining the *derived category* of an abelian category, whose objects are still complexes, but where all quasi-isomorphisms become isomorphisms (see quotes from [10] below). He described these ideas to Verdier and proposed them to him as a thesis subject. Verdier soon developed the basic theory, introducing triangulated categories along the way [18].

I met Grothendieck when he came to Harvard in the fall of 1961. I wrote lecture notes for his seminar on Local Cohomology [8]. I wrote my thesis [12] on the structure of the Hilbert scheme, which he had described in his seminar

Construction Techniques the same fall. In the spring of 1963 I had just finished writing up my thesis, and was offered a three year research fellowship at Harvard. I was looking for a new project. In order to participate in Grothendieck's new way of developing algebraic geometry, I offered to give a seminar on his theory of duality, a vast extension of the earlier "Serre duality," which had not yet appeared. He agreed, saying in his letter to me of 4.29.63:

> I would be glad if you could run a seminar on duality for coherent sheaves, as I have this on my mind for quite a few years without finding time to run a seminar about it myself. I am ready to send you a detailed outline of the theory, starting some time in July (for the moment I am too busy by my own seminar and various things). Of course you would have to fill in many details, which would involve a good deal of work. Certainly it would be easier (but less useful) to run your baby-seminar on the SGA 1962 stuff, as there are detailed seminar notes available [9]. I was more or less determined to publish somewhere a sketch of the duality theory (both in the algebraic and the complex-analytic set-up) without proof, referring for these to EGA Chap 10, to appear in 1970. Of course it would be much nicer to have sound seminar notes on the subject, available around 1964. Please tell me if you feel courage enough to run a seminar on this rather messy subject, and to *write notes* to make it available to other people, as otherwise it would be of much more restricted interest.

I said yes, and by August 1963 he had nearly completed a 250 page manuscript *Résidus et Dualité; prénotes pour un Séminaire Hartshorne* (these prenotes). I gave lectures on this material at Harvard in the fall and winter of 1963/64, aided by Mumford, Tate, Lichtenbaum, Fogarty, and others, and wrote at the time 6 exposés, sending each one to Grothendieck for commentary. Later I rewrote and completed these into the book *Residues and Duality* [11].

The argument

The theory of duality as presented in the prenotes is a vast expansion and generalization of the earlier theory. The goal now is no longer just a duality for projective varieties over a field. It becomes a duality for an arbitrary proper morphism of schemes, stated and proved in the language of derived categories. To break away from excessive dependence on projective space (though it does reappear near the end in the form of Chow's lemma), the theory includes a local theory of duality and a local construction of the residual complex. To give sufficient flexibility to the theory, everything must be functorial, and there are millions of functorial compatibilities to check along the way.

Here is what the ideal duality theorem would look like. For a scheme X, denote by $D(X)$ the derived category of the category of $\mathcal{O}_X$-modules. For a functor on sheaves, e.g. f_* where $f : X \to Y$ is a morphism of schemes, we denote by $Rf_* : D^+(X) \to D^+(Y)$ the corresponding right derived functor on the derived categories of bounded below complexes. Then, for a proper morphism of schemes, we seek

a) a functor $f^! : D^+(Y) \to D^+(X)$ and
b) a trace map $\mathrm{Tr}_f : Rf_* f^! \to \mathrm{Id}_Y$, such that
c) the induced map

$$R\mathrm{Hom}_X(F, f^! G) \to R\mathrm{Hom}_Y(Rf_* F, G)$$

is an isomorphism for all complexes of sheaves F in $D^+(X)$ and G in $D^+(Y)$.

For the case of a smooth morphism, we can take $f^! G = f^* G \otimes \Omega^n_{X/Y}[n]$, where n is the relative dimension. Then for example if F is a single sheaf and $G = \mathcal{O}_Y$, this theorem says that

$$R\mathrm{Hom}_X(F, \Omega^n_{X/Y}[n]) \to R\mathrm{Hom}_Y(Rf_* F, \mathcal{O}_Y)$$

is an isomorphism.

Specializing further to the case Y is the spectrum of a field, the complex on the left has cohomology groups $H^{-i} = \mathrm{Ext}^{n-i}_X(F, \Omega^n_{X/Y})$ and on the right we have simply the dual (over k) of $H^i(X, F)$. Thus we recover the old form of duality.

I should remark immediately that one does not expect the theorem as stated above to be true without some restrictive hypotheses. It can be proved, for example, if $f : X \to Y$ is a proper morphism of schemes of finite type over a field, taking F to be a complex bounded above, with quasi-coherent cohomology sheaves, and G a complex bounded below, with coherent cohomology sheaves.

The subtlest part of the theory is the construction of the functor $f^!$. This must be done locally on the source X and hence for (not necessarily proper) morphisms of finite type. For a smooth morphism $f : X \to Y$ of relative dimension n, one can take $f^! = f^* \otimes \Omega^n_{X/Y}[n]$. For a finite morphism one can take $f^! = R\mathcal{H}om_Y(f_* \mathcal{O}_X, -)|X$. Thus by taking open affines, we can construct $f^!$ locally on an open cover of X. The difficulty is that for G a complex on Y, we get $f^! G$ in the derived categories $D(U_i)$ for U_i open in X. These constructions naturally are isomorphic on overlaps, but objects of the derived categories $D(U_i)$ for an open covering $\{U_i\}$ of X, together with patching data on the overlaps, may not glue to give an element of $D(X)$. This is an inevitable drawback of working with derived categories.

At this point in the prenotes, Grothendieck starts developing a general theory of *pseudo-complexes* consisting of objects of the derived category of a covering of a scheme, with patching data. In writing [11], to keep my own labor within bounds, and finish the project in a finite amount of time, I ruthlessly imposed restrictive hypotheses so that abstractions like pseudo-complexes would not be necessary. Cf. [11, VI.3] where I deal only with morphisms of finite type of locally noetherian schemes with bounded dimension of the fibers.

To complete the definition of the functor $f^!$, Grothendieck defines a residual complex on a scheme, which may or may not exist in general, and is not unique, but is an actual complex. If one has a residual complex K_Y on Y, then $f^! K_Y$ has the remarkable property that there is an actual complex K_X associated to it functorially, and K_X is then a residual complex on X. Being actual complexes, these residual complexes, defined on an open covering of X, glue to give a residual complex on X. Furthermore, the last piece of the residual complex is a direct sum of injective hulls of residual fields $k(x)$ over local rings $\mathcal{O}_{X,x}$ for points x that are finite over their images in Y, and for these a generalized residue theorem allows one to define the trace map for residual complexes.

Now the major hurdles are overcome: one has $f^!$ and Tr_f defined compatibly for all morphisms and so also one has the duality morphism. To prove it is an isomorphism, one uses Chow's lemma to reduce to the projective case proved earlier.

As Verdier points out in the introduction to his thesis [19], even to prove the duality theorem in the case of a proper smooth variety X over an algebraically closed field, and for locally free sheaves, there is no known proof other than the one just given passing through the general theory with arbitrary schemes and derived categories.

A retrospective view

About twenty years after writing these prenotes, and well after the end of his intense public activity in the area of algebraic geometry, Grothendieck wrote a long rambling essay about mathematics, his work, and his colleagues, called *Récoltes et Semailles* [10]. In it one can find his comments on how he came to the theorems on duality both in the case of coherent sheaves on schemes and in other cases such as étale cohomology, together with his guiding philosophy in developing the theory. Here are some excerpts.

> [10, p. 42] Parmi les nombreux points de vue nouveaux que j'ai dégagés en mathématique, il en est **douze**, avec le recul, que j'appellerais des « grandes idées. » [Number two in the list is] Dualité « continue » et « discrète » (catégories dérivées, « six opérations »).

[10, p. 186] L'autre texte est une esquisse d'un « **formulaire des six variances** » (1), rassemblant les traits communs à un formalisme de dualité (inspiré de la dualité de Poincaré et de celle de Serre) que j'avais dégagé entre 1956 et 1963, formulaire qui s'est avéré avoir un caractère « universel » pour toutes les situations de dualité cohomologique rencontrées à ce jour.

[10, p. 415] Il s'agit en premier lieu de l'idée de **catégorie dérivée** en algèbre homologique [...], et de son utilisation pour un formalisme « passe-partout », dit « **formalisme des six opérations** » (à savoir les opérations

$$\otimes^L,\ Lf^*,\ Rf_!,\ R\mathcal{H}om,\ Rf_*,\ f^!$$

pour la cohomologie des types d'« espaces » les plus importants qui se sont introduits jusqu'à présent en géométrie). [. . .]
La découverte progressive de ce formalisme de dualité et de son ubiquité s'est faite par une réflexion solitaire, obstinée et exigeante, qui s'est poursuivie entre les années 1956 et 1963. C'est au cours de cette réflexion que s'est dégagée progressivement la notion de catégorie dérivée, et une compréhension du rôle qui lui revenait en algèbre homologique.

[10, p. 427] Un premier pas vers une compréhension approfondie de la dualité en cohomologie a été la découverte progressive du formalisme des six variances dans un premier cas important, celui des schémas noethériens et des complexes de modules à cohomologie cohérente. Un deuxième a été la découverte (dans le contexte de la cohomologie étale des schémas) que ce formalisme s'appliquait également pour des coefficients discrets. Ces deux cas extrêmes étaient suffisants pour fonder la conviction de l'**ubiquité** de ce formalisme dans toutes les situations géométriques donnant lieu à une « dualité » du type Poincaré – conviction qui a été confirmée par les travaux (entre autres) de Verdier, Ramis et Ruget.

[10, p. 439] Comme il est bien connu, la théorie des catégories dérivées est due à J.-L. Verdier. Avant qu'il entreprenne le travail de fondements que je lui avais proposé, je m'étais borné à travailler avec des catégories dérivées de façon heuristique, avec une définition provisoire de ces catégories (qui s'est avérée par la suite être la bonne), et avec une intuition également provisoire de leur structure interne essentielle (intuition qui s'est révélée techniquement fausse [...]). La théorie de dualité des faisceaux cohérents (i.e. le formalisme des « six variances » dans le cadre cohérent) que j'avais développée vers la fin des années 1950 ne prenait tout son sens que *modulo* un travail de fondements sur la notion de catégorie dérivée qui a été fait par Verdier ultérieurement.

[10, p. 587] Vers l'année 1960 ou 1961 je propose à Verdier, comme travail de thèse possible, le développement de nouveaux fondements de l'algèbre homologique, basé sur le formalisme des catégories dérivées que j'avais dégagé et utilisé au cours des années précédentes pour les besoins d'un formalisme de dualité cohérente dans le contexte des schémas. [...] Verdier accepte le sujet proposé. Son travail de fondements se poursuit de façon satisfaisante, se matérialisant en 1963 par un « état 0 » sur les catégories dérivées et triangulées, multigraphié par les soins de l'IHÉS. C'est un texte de cinquante pages, reproduit en Appendice à SGA $4\frac{1}{2}$ en 1977 [18].

[10, p. 1512] Dans la seconde moitié des années 1960, j'avais développé dans le contexte des schémas un formalisme de « dualité cohérente ». Ces réflexions, motivées par le désir

(1) These are the same as the « six opérations » in [10], p. 415 below.

> de comprendre le sens et la portée exacte du théorème de dualité de Serre en géométrie analytique et surtout en géométrie algébrique, avaient été poursuivies dans une solitude à peu près complète, n'ayant pas l'air d'intéresser personne d'autre que moi. [And in footnote 268] Serre a toujours refusé d'écouter [...]. Je crois que je n'ai guère essayé d'en parler à quiconque d'autre, mis à part (bien plus tard) Hartshorne, qui a fait sur mes idées un très beau séminaire à Harvard, publié en 1966 [11].
>
> [10, p. 1547] Mais je sais bien, quant à moi, qu'avec les conjectures de Weil et avec l'intuition omniprésente des topos, la vision des six opérations a été ma principale source d'inspiration dans mes réflexions cohomologiques tout au long des années 1955–1970.

What we can see from these reflections is that the philosophy of "les six opérations," to which Grothendieck attaches so much importance in [10], developed gradually over time and reached its full realization only after many years. In fact, the collective term "six opérations" appears for the first time in [10], and not in any of the SGA seminars. One can perceive the seeds of this concept in the present prenotes, both in the existence of so many compatibilities between the various functors, and also in the necessity of this approach for the definition of the functor $f^!$. Indeed, since the construction of $f^!$ is built out of one method for smooth morphisms and another method for finite morphisms, in order to glue together the results on open sets, one needs full control of all the relationships among them.

For information about how these ideas have led to more duality theorems in other contexts, such as étale cohomology, we refer the reader to the introduction to Verdier's thesis [19] and the Commentaires of Grothendieck at the beginning of these prenotes.

Notes on the text

This text was written in 1963 and uses terminology that was current at the time. Hence there are some words that have been replaced by others in recent times. For example: "préschéma" is now called "schéma," "morphisme simple" is now called "morphisme lisse".

Chapter 1 of the original prenotes was the text of Verdier [18]. This was an original typescript, with author's corrections overtyped with xxx, and with special symbols and arrows drawn in by hand. Since this work has been published elsewhere, it is not included here. The published version has been retyped to clean up corrections, but the text is almost identical to the one used here. It contains the theory of triangulated categories, derived categories, and derived functors. This material is reproduced in [11, Ch. I] except that the terminology "way out functors" was my addition.

Chapter 2 contains the duality theorem for a projective space morphism $\mathbf{P}^n_S \to S$. This result, for complexes of quasi-coherent sheaves, follows almost immediately from the explicit calculations of cohomology in [7], brought up to speed in the language of derived categories. This chapter also contains a duality theorem for a morphism $f : X \to S$ that factors through a projective space morphism. Here one needs a functor $i^!$ for a closed immersion i. Grothendieck refers to par. 1 for this result (p. 13), but unfortunately it was not included in Verdier's summary. One can use the corresponding functor for a finite morphism, dealt with later [11, III.6].

Chapter 3 contains the statement of the general duality theorem for a smooth proper morphism. It is easy to state, because in this case the functor $f^!$ is evident—it is simply $f^* \otimes \Omega^n_{X/S}[n]$. However, at this point in the prenotes, one can only prove the theorem in case the morphism factors through a projective space morphism. This material is treated in [11, Ch. III]. There are remarks scattered through Chapters 2 and 3 about how to possibly extend all these results to the non-noetherian case for not necessarily quasi-coherent sheaves. There are also asides about the possible use of fibered categories, and a section posing questions about extending the duality theorem to include differential operators. Grothendieck wanted to treat each part of his work in maximum generality to anticipate future developments and because he believed that a theorem without restrictive hypotheses led to a better understanding.

Chapter 4 contains the conjectural existence theorems of $f^!$ and Tr_f for morphisms locally of finite type, together with all the desired compatibility properties. From these conjectural theorems, the duality theorem follows formally. Grothendieck says, at the beginning of Chapter 4, that they will be proved later assuming the schemes are of finite type over a regular noetherian scheme of finite dimension, and that the complexes have coherent cohomology. "Bien entendu, ce sont là des restrictions artificielles qui devraient être éliminées" (p. 35).

In Chapter 5, Grothendieck begins preparing the way for the general duality theorem. He shows how to construct the functor $f^!$ for a morphism of affine schemes, so that the general construction becomes a question of gluing. But elements of the derived category on open sets with glueing data in general do not glue. So this chapter is devoted to the theory of pseudo-complexes, which Grothendieck describes as an indispensable intermediary (Remarques 5.2.1) for the general construction of $f^!$. A *pseudo-complex* on X is given by an open covering $\{U_i\}$ of X together with elements of the derived categories $D(U_i)$ and gluing data in the derived categories of the overlaps. The pseudo-complexes do not form a triangulated category, but many of the operations on derived categories carry over to them. In particular, for a morphism $f : X \to Y$ one can define $f^!$ of a pseudo-complex on Y as a pseudo-complex on X.

The rest of Chapter 5 is devoted to topics not of direct relevance to the proof of the duality theorem, so they mostly do not appear in [11]. These include the study of the fundamental module for a morphism, which in case of a relatively Cohen-Macaulay or relatively Gorenstein morphism is essentially a single sheaf. There is a detailed study of trace and residue maps for the fundamental module. There is also Cartier's construction of the residue map, and a formulary for computing the residue symbol. There is also a study of the *cotangent complex* of a morphism, and the theory of the *different.*

Chapter 6 is all about local cohomology and resolutions. It is independent of what has come before, and much of it holds on arbitrary topological spaces. The results on schemes will be used in later chapters to construct the residual complex.

The treatment here goes beyond what is in [8] and [9], in that Grothendieck considers not only closed supports, but also families of supports, sheaves of families of supports, and a filtration of a topological space by a descending sequence of sheaves of families of supports. For each of these he defines the local cohomology of an abelian sheaf or a complex of sheaves. Most interesting for the sequel is a complex canonically associated to a sheaf, obtained by taking its relative local cohomology sheaves on one step of a filtration modulo the next (Proposition 6.4.4). On a scheme it is called the *Cousin complex* of the sheaf. In the case of a regular scheme, the Cousin complex of the structure sheaf becomes a resolution, while as a module, it is just the direct sum of injective hulls of the points over their local rings (Théorème 6.7.2). This is the model for the residual complexes considered later.

Chapter 7 deals with dualizing complexes, local duality, and residual complexes. A dualizing complex R is an element of the derived category $D(X)$ such that the functor $D = R\mathcal{H}om(-, R)$ gives an autoduality (i.e., D^2 is the identity) on the category $D(X)$. Grothendieck explains the relation between a dualizing complex on X and the local duality theorem for the local rings of X. Then he defines the notion of residual complex on X: it is an actual complex of $\mathcal{O}_X$-modules such that as an object of the derived category it is a dualizing complex, while as a module it is the direct sum of the injective hulls of the residue fields of the points of X. He shows that the Cousin complex of a dualizing complex is a residual complex. Now comes the tour de force (Section 7.5): if $f : X \to Y$ is a morphism, and if R_Y is a residual complex on Y, then its $f^!$ is a dualizing pseudo-complex on X. Since the Cousin complex construction is functorial, the associated residual complexes on open subsets of X glue together to give a global residual complex on X. From here the general construction of $f^!$ follows easily.

In Chapter 8, Grothendieck pulls everything together, constructing the trace map for residual complexes, and proving that it is a morphism of complexes for a proper morphism (Théorème 8.1.1). This step of the proof requires a reduction to the classical result that the sum of the residues is zero for a differential form on a smooth proper curve over a field. Then comes the general duality theorem, proved here for a proper morphism of schemes that are noetherian, of finite dimension, admitting a residual complex, and complexes with coherent cohomology (Théorème 8.3.3).

The material of Chapters 6 and 7 on local cohomology, filtrations, the Cousin complex, the dualizing complex, the local duality theorem, and the definition of residual complexes appears in a similar form in [11, Ch. IV, V, and VI]. However, the construction of $f^!$ for residual complexes is proved directly in [11, VI.3], without passing through the pseudo-complexes. After that, the proof of the general duality theorem in [11] is the same as here.

Chapter 9, on the cohomology class associated to a cycle, generalizing the treatment in [5] of a simpler case, was never written. Grothendieck gave me an oral exposé of this material in May 1965, of which I took notes, but I never worked them out, and the notes are incomprehensible to me today.