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ON THE p-ADIC UNIFORMIZATION OF UNITARY SHIMURA CURVES

Stephen Kudla, Michael Rapoport, Thomas Zink

Abstract. – We prove p-adic uniformization for Shimura curves attached to the group of unitary similitudes of certain binary skew Hermitian spaces V with respect to an arbitrary CM field K with maximal totally real subfield F. For a place $v|p$ of F that is not split in K and for which V_v is anisotropic, let ν be an extension of v to the reflex field E. We define an integral model of the corresponding Shimura curve over $\operatorname{Spec} O_{E,(\nu)}$ by means of a moduli problem for abelian schemes with suitable polarization and level structure prime to p. The formulation of the moduli problem involves a *Kottwitz condition*, an *Eisenstein condition*, and an *adjusted invariant*. The uniformization of the formal completion of this model along its special fiber is given in terms of the formal Drinfeld upper half plane $\widehat{\Omega}_{F_v}$ for F_v. The proof relies on the construction of the *contracting functor* which relates a relative Rapoport-Zink space for strict formal O_{F_v}-modules with a Rapoport-Zink space of p-divisible groups which arise from the moduli problem, where the O_{F_v}-action is usually not strict when $F_v \neq \mathbb{Q}_p$. Our main tool is the theory of displays, in particular the *Ahsendorf functor*.

Résumé. – On démontre l'uniformisation p-adique pour les courbes de Shimura attaché à un groupe de similitudes unitaires pour certains espaces anti-hermitiens V relatifs à un corps CM K, avec sous-corps totalement réel maximal F. Pour une place $v|p$ de F qui n'est pas déploye dans K et pour laquelle la localisation V_v est anisotrope, soit ν une extension de v au corps reflex E. On définit un modèle sur $\operatorname{Spec} O_{E,(\nu)}$ de la courbe de Shimura correspondante en posant un problème de modules de variétés abéliennes avec polarisation et structure de niveau premier à p. La formulation du problème de modules fait intervenir une *condition de Kottwitz*, une *condition d'Eisenstein*, et la notion d'un *invariant rectifié*. L'uniformisation du complété formel de ce modèle le long sa fibre spéciale est donné en termes du démi-plan de Drinfeld formel $\widehat{\Omega}_{F_v}$ pour F_v. La démonstration est basée sur la construction d'un *foncteur contractant* qui rélie un espace de Rapoport-Zink relatif de O_{F_v}-modules formels stricts avec un espace de Rapoport-Zink de groupes p-divisibles des variétés abéliennes qui apparaissent dans le problème de modules, pour lesquelles l'action de

O_{F_v} n'est pas stricte en géneral si $F_v \neq \mathbb{Q}_p$. Notre outil principal est la *théorie des displays*, en particulier le *foncteur de Ahsendorf*.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. History of uniformization

One of the major results of the Mathematics of the 19th century is the *uniformization theorem.* It states that any non-singular projective algebraic curve X of genus $g(X) \geq 2$ can be uniformized, i.e., can be written as

$$X \simeq \Gamma \backslash \Omega_{\mathbb{R}}, \tag{1.1.1}$$

where $\Omega_{\mathbb{R}} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is the union of the upper and the lower half plane and Γ denotes a discrete cocompact subgroup of $\mathrm{PGL}_2(\mathbb{R})$. This notation reinforces the analogy with the p-adic uniformization discussed below. The history of this theorem is very complicated, and involves the names of many mathematicians, among them Poincaré, Hilbert and Koebe, comp. [**13**]. Inspired by the uniformization theorem, Poincaré gave a systematic construction of cocompact discrete subgroups of $\mathrm{PGL}_2(\mathbb{R})$. For this he used the exceptional isomorphism between inner forms of PGL_2 and special orthogonal groups of ternary quadratic forms. In fact, for his construction, he used arithmetic subgroups of the special orthogonal group of an indefinite anisotropic ternary quadratic form over $\mathbb{Q}$, cf. [**13**].

Now let p be a prime number. The history of the p-adic uniformization of algebraic curves starts with Tate's uniformization theory of elliptic curves. It turns out that not all elliptic curves over p-adic fields admit a p-adic uniformization, but only those with (split) multiplicative reduction [**30**, §6].

The next step was Mumford's p-adic uniformization theory of algebraic curves of higher genus, [**24**]. Again, it turns out that not all such algebraic curves over p-adic fields admit a p-adic uniformization, but only those with totally degenerate reduction [**24**]. In view of Mumford's results, it becomes interesting to single out classes of algebraic curves with totally degenerate reduction. Such classes are exhibited by Cherednik [**7**].

Cherednik's discovery is that certain *quaternionic Shimura curves*, i.e., Shimura curves associated to quaternion algebras over a totally real field F, admit p-adic uniformization. The quaternion algebra has to satisfy the following conditions. It is required to be split at precisely one archimedean place w of F (and ramified at all other archimedean places), and to be ramified at a non-archimedean place v of residue characteristic p. In this case, the reflex field can be identified with F. Then one obtains p-adic uniformization by the Drinfeld halfplane associated to F_v, provided that the level structure is prime to v. It follows that if X is a connected component of the Shimura tower for such a level, considered as an algebraic curve over $\bar{F}$, then there is an isomorphism of algebraic curves over $\bar{F}_v$,

$$X \otimes_{\bar{F}} \bar{F}_v \simeq (\bar{\Gamma} \backslash \Omega_{F_v}) \otimes_{F_v} \bar{F}_v. \tag{1.1.2}$$

Here $\Omega_{F_v} = \mathbb{P}^1_{F_v} \setminus \mathbb{P}^1(F_v)$ denotes the Drinfeld halfplane for the local field F_v, and $\bar{\Gamma}$ denotes a discrete cocompact subgroup of $\mathrm{PGL}_2(F_v)$. Recall that Ω_{F_v} is a rigid-analytic space over F_v. The isomorphism (1.1.2) is to be interpreted as follows: the rigid-analytic space $\bar{\Gamma} \backslash \Omega_{F_v}$ is (uniquely) algebraizable by a projective algebraic curve over F_v. After extension of scalars $F_v \longrightarrow \bar{F}_v$, there exists an isomorphism as in (1.1.2). We thus see that (1.1.2) allows us to pass from the original complex uniformization $X \otimes_{\bar{F}} \mathbb{C} \simeq \Gamma \backslash \Omega_{\mathbb{R}}$, where Γ is a congruence subgroup maximal at v, to p-adic uniformization.

Let us comment on the proof of Cherednik's theorem. When $F = \mathbb{Q}$, these quaternionic Shimura curves are moduli spaces of abelian varieties with additional structure, and Drinfeld **[10]** gives a moduli-theoretic proof of Cherednik's theorem in this special case. Furthermore, he proves an 'integral version' of this theorem (which has the original version as a corollary). For this, Drinfeld extends the moduli problem integrally and then relates the integral version to a theorem on formal moduli spaces of p-divisible groups, which is in fact the deepest part of Drinfeld's paper. When $F \neq \mathbb{Q}$, Cherednik's quaternionic Shimura curves do not represent a moduli problem of abelian varieties, and Drinfeld's approach runs into problems. Cherednik's approach **[7]** seems to only use arguments involving the generic fiber.

There are also higher-dimensional versions of p-adic uniformization. Drinfeld's method has been generalized by Rapoport and Zink **[27]** to Shimura varieties associated to certain *fake unitary groups*. These are associated to central division algebras over a CM-field equipped with an involution of the second kind; for Rapoport-Zink uniformization, one has to assume that the p-adic place of the totally real subfield splits in the CM-field. This higher-dimensional generalization also includes integral uniformization theorems. In **[27]**, these integral uniformization theorems appear as a special instance of a general *non-archimedean uniformization theorem*, which describes the formal completion of PEL-type Shimura varieties along a fixed isogeny class. In the case of p-adic uniformization, the whole special fiber forms a single isogeny class.

The method of **[27]** has been applied by Boutot and Zink **[5]** to prove Cherednik's original theorem and an integral variant of it by embedding Cherednik's quaternionic Shimura curves into Shimura curves obtained by the Rapoport-Zink method; in an update **[6]**, some gaps in **[5]** are filled. The integral uniformization theorems in **[6]** have the draw-back that they only show that there exists some integral model of the Shimura curve for which one has integral uniformization. There is a characterization of this integral model as the unique stable model in the sense of Deligne-Mumford **[9]** but