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EVOLUTION OF NETWORKS WITH MULTIPLE JUNCTIONS

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EVOLUTION OF NETWORKS WITH MULTIPLE JUNCTIONS

by Carlo MANTEGAZZA, Matteo NOVAGA,
Alessandra PLUDA & Felix SCHULZE

Abstract. — We consider the motion by curvature of a network of curves in the plane and we discuss existence, uniqueness, singularity formation, and asymptotic behavior of the flow.

Résumé. (**Évolution des réseaux à jonctions multiples**) — On considère le mouvement par courbure d'un réseau de courbes dans le plan et on discute de l'existence, l'unicité, la formation des singularités et le comportement asymptotique du flux.

CONTENTS

CHAPTER 1

INTRODUCTION

In this work we give an overview of the state-of-the-art of the motion by curvature of planar networks of curves, collecting known results and showing several new ones.

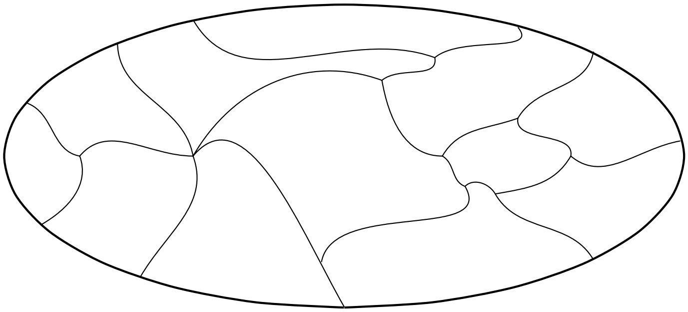

Figure 1.1: A planar network of curves in a convex domain.

The problem, proposed by Mullins [12] and discussed first in [12, 16, 17, 47, 63], attracted the interest of many authors in recent years [10, 15, 21, 34, 42, 51, 58, 62, 76, 77, 80, 82, 83, 89, 94–97, 106]. One strong motivation to study this flow is the analysis of models of two–dimensional multiphase systems, where the problem of the dynamics of the interfaces between different phases arises naturally. As an example, the model where the energy of a configuration is simply given by the total length of the interfaces has proven useful to describe the growth of grain boundaries in a polycrystalline material (see [12, 47, 63] and `http://mimp.materials.cmu.edu`).
A second motivation is more theoretical: the evolution by curvature of such a network of curves is the simplest example of mean curvature flow of a set which is *essentially* singular. To consider such flow not only for smooth submanifolds but also for non–regular sets, several generalized (weak) definitions of the flow have been introduced in the literature [2, 16, 25, 35, 56, 101]. Anyway, while the smooth case was

largely studied and understood (even if still not completely), the evolution of generalized submanifolds, possibly singular (for instance *varifolds*), has not been analyzed in great detail.
In his seminal paper, K. Brakke [16] proved the existence of a global (very) weak solution, in a *geometric measure theory* context, called "Brakke flow". Recently, the work of Brakke has been improved by L. Kim and Y. Tonegawa [62] (see also [105]) in the case of the evolution of grain boundaries in $\mathbb{R}^n$ (which reduces to the evolution of networks when $n = 2$). They proved a global existence theorem and also showed that there exists a finite family of open sets moving continuously with respect to the Lebesgue measure, whose boundaries coincide with the space–time support of the flow (for further results, see also the papers by K. Kasai and Y. Tonegawa [60] and Y. Tonegawa and N. Wickramasekera [106]). Finally, in [61], Kim and Tonegawa also proved a regularity result for 1–dimensional Brakke flows, showing that for almost all times, the evolving network consists of a finite number of embedded curves of class $W^{2,2}$, meeting at junctions with angles of 60 or 120 degrees or with a common tangent.
For another global existence result in any codimension and with special regularity properties, obtained adapting the elliptic regularization scheme of T. Ilmanen [55, 56], we refer to the work of the last author and B. White [98]. Despite these recent improvements, Brakke's definition is anyway apparently too weak (possibly too general) if one is interested in a detailed description of the flow.
A completely different definition of evolution is instead based on the so-called *minimizing movements*: an implicit time–discrete variational scheme introduced in [2, 71] (see also [14, 18, 26]). In this context, another discretization scheme was developed and studied by S. Esedoglu and F. Otto [34], T. Laux and F. Otto [68, 69] (we motion also the more recent development [36]).
Finally, we mention the "level set" approach to motion by curvature by L. C. Evans and J. Spruck [35] or, alternatively, Y. G. Chen, Y. Giga, and S. Goto [22], unfortunately not suitable for the motion of networks since if at least a multi–point is present then an interior region immediately develops (the so-called "fattening" phenomenon).
Even if all these approaches provide a globally defined evolution, the possible conclusions on the structure and regularity of the moving networks are actually quite weak. To obtain a detailed description of the evolution and of the singularity formation, we tried to work in the smooth setting as much as possible. The definition of the flow is then the first problem one has to face, due to the contrast between such desire and the intrinsic singular geometric nature of a network. Consider for instance the network described by two curves crossing each other, forming a 4–point. There are actually several possible candidates for the flow: one cannot easily decide how the angles must behave, moreover, it could also be allowed the four concurrent curves to separate into two pairs of curves moving independently of each other and/or we

could take into account the possible "birth" of new multi–points from such a single one (all these choices are possible with Brakke's definition). Actually, one would like a good/robust definition of curvature flow giving uniqueness of the motion (at least for "generic" initial networks) and forcing the evolving network, by an "instantaneous regularization" effect, with the possible exception of some discrete set of times, to have only triple junctions with the three angles between the concurring curves of 120 degrees. This last property (which was experimentally observed for the growth of grain boundaries) is usually called *Herring condition*. These expectations are sustained also by the variational nature of the problem since this evolution can be considered as the "gradient flow" in the "space of networks" of the *Length* functional, which is the sum of the lengths of all the curves of the network (see [16]). It must anyway be said that such a space does not share a natural linear structure and such a "gradient" is not actually a well-defined "velocity" vector driving the motion at the multiple junctions, in general. However, it follows that every point of a network different from its multi–points must move with a velocity whose normal component is the curvature vector of the curve it belongs, in order to decrease the *Energy* of the network (that is, the total length here) "most efficiently" (see [16]). From this "energetic" point of view, it is then natural to expect also that configurations with multi–points of order greater than three or 3–points with angles different from 120 degrees, being *unstable* for the length functional, should be present only in the initial network or that they should appear only at some discrete set of times, during the flow. This property is suggested also by numerical simulations and physical experiments, see [12, 17, 47, 63] and the *grain growth* movies at `http://facstaff.susqu.edu/brakke`. One may hope that some sort of "parabolic regularization" could play a role here: for instance, if a multi–point has only two concurrent curves, it can be easily shown (see [4, 6, 7, 46]) that the two curves become instantaneously a single smooth curve moving by curvature.

We mention that actually, it is always possible to find a Brakke flow sharing such property at almost every time (see [16]), by the variational spirit of its definition which is the closest to the "gradient flow" point of view. However, as uniqueness does not hold in this class, there are also Brakke flows starting from the same initial network which keeps their multi–points, or loose the connectedness of the network: for instance, a 4–point can "open" as in the right side of Figure 12.1, or separate in two no more concurring curves, or it can "persists" to be a 4–point where the two "crossing" curves move independently. Anyway, as we said, Brakke's definition is too "weak" if one is interested in a detailed description of the flow.

By this discussion it is then natural, due to their expected relevance, to call *regular* the networks with only 3–points and where the three concurrent curves form angles of 120 degrees. Then, following the "energetic" and experimental motivations mentioned above, we simply *impose* such regularity condition in the definition of a *smooth*

curvature flow, for every positive time (at the initial time it could fail). If the initial network is regular and smooth enough, we will see that this definition leads to an almost satisfactory (in a way "classical") short–time existence theorem of a flow by curvature. Trying instead to let evolve an initial non–regular network, various complications arise related to the presence of multi–points or of 3–points not satisfying the Herring condition. Notice also that, even starting with an initial regular network, we cannot avoid to deal also with non–regular networks when we analyze the global behavior of the flow. Indeed, during the flow, some of the triple junctions could "collide" along a "vanishing" curve of the network, when the length of the latter goes to zero (hence, modifying the topological structure of the network). In this case one has to "restart" the evolution with a different set of curves, possibly describing a non–regular network, typically with multi–points of order higher than three (consider, for instance, two 3–points collapsing along a single curve connecting them) or even with "bad" 3–points with angles between the concurring curves, not all equal to 120 degrees (think of three 3–points collapsing together with the "triangular" region delimited by three curves connecting them). A suitable short–time existence (hence, "restarting") result for this situation has been worked out in [58] by T. Ilmanen, A. Neves and the fourth author and in [70] by J. Lira, R. Mazzeo, M. Saez and the third author. In these papers, it is indeed shown that starting from any non–regular network (with a natural technical hypothesis), there exists a "satisfactory" flow of networks by curvature which is immediately regular and smooth, for every positive time. Chapter 11 is devoted to this topic.
The existence problem of a curvature flow for a regular network with only one 3–point and fixed end–points, called *triod* (see Definition 3.2), was first considered by L. Bronsard and F. Reitich in [17]. To be precise, they consider as initial datum any regular $C^{2+2\alpha}$ triod satisfying some compatibility conditions at the triple junctions and show short–time existence and uniqueness in the parabolic class $C^{2+2\alpha,1+\alpha}$. In [63] D. Kinderlehrer and C. Liu proved the global existence and convergence of a smooth solution if the initial regular triod is sufficiently close to a minimal (Steiner) configuration.
After introducing regular networks, their flow by curvature, and some basic properties (Sections 2 and 2.3), we extend, in Chapter 3, the above well–posedness theorem to general regular networks (Theorem 3.25). Moreover, we also show an analogous result in suitable Sobolev spaces (Theorem 3.6).
In Chapter 4 we generalize to any regular network the integral estimates proved in [82] for a triod, which are needed to prove Theorem 5.8 and will be actually used throughout the whole paper. A consequence of such estimates is the fact that if the lengths of the curves are bounded away from zero, as t goes to the maximal time T of existence of the flow, the maximum of the modulus of the curvature must go to $+\infty$ (Corollary 4.15 and Theorem 5.7).

The uniqueness of the flow is quite delicate. Indeed, by Theorem 3.25, we only have that, for initial regular networks of class $C^{2+2\alpha}$ having the sum of the curvatures of the three concurring curves at every triple junction equal to zero, there is uniqueness in the parabolic class $C^{2+2\alpha,1+\alpha}$. In Chapter 5, by combining Theorems 3.6 and 3.25 (the first mainly for the uniqueness, the second for the existence) we then show a result of existence/geometric uniqueness for short time of the flow of an initial network of class C^2 (Theorem 5.8), in a subclass of the curvature flows which are simply C^2 in space and C^1 in time. In the same section, we will also see that the classical property of parabolic equations of instantaneous regularization of solutions for positive times also holds for the motion by curvature of networks, in a suitable sense.
The rest of the paper is devoted to the long-time behavior of the flow. For the sake of simplicity, in the following overview, we will restrict ourselves only to the behavior in the interior of a convex domain of a network flowing by curvature with fixed end–points on the boundary of such set, while in the whole paper also the behavior at the boundary (hence, at the end–points of the network) is analyzed in the same detail.
In Chapter 7 we recall Huisken's monotonicity formula for mean curvature flow which holds also for the evolution of a network and we introduce the rescaling procedures to get blow–up limit networks (discussed in Chapter 8) at the maximal time of smooth existence. Then, to "describe" the singularities of the flow one needs to classify such possible blow–up limits. In some cases, arguing by contradiction with geometric arguments, this "description" can be used to exclude at all the formation of singularities. Key references for this method in the situation of a single smooth closed curve are [3, 50, 52, 53]. The most relevant difference in dealing with networks is the difficulty in using the maximum principle, which in the case of closed curves is the main tool for getting pointwise estimates on the geometric quantities during the flow. For this reason, some crucial estimates which are straightforward in such case are here much more difficult to obtain and we had to resort to the integral estimates of Chapter 4 (see also Section 10.3), which are similar to the ones in [3, 6, 7, 54], but require some extra work to deal with the triple junctions.
One can reasonably expect that an embedded regular network does not develop singularities during the flow if its "topological structure" does not change (for instance, in the case of a "collision" of two or more 3–points). Our analysis in Sections 8, 9 and 10 will show that if no "multiplicities" larger than one occur in the blow–up limit networks, this expectation is indeed true. Under the assumption that the lengths of the curves are bounded away from zero the only possible blow–up limits (with multiplicity one by hypothesis) are either a straight line, a halfline, or a flat unbounded regular triod (called "standard triod") composed of three halflines through the origin of $\mathbb{R}^2$ forming angles of 120 degrees (see Proposition 8.30 and Chapter 10). Then, a local regularity theorem for the flow (shown in [58]) together with such classification excludes the presence of singularities. This result, which is in the spirit of White's local regularity theorem for mean curvature flow in [110], is presented in detail in Chapter 9.

Thus, again in Chapter 10, we try to understand what happens at the maximal time, knowing that some lengths of the curves composing the network cannot be uniformly bounded away from zero, hence at least two 3–points get closer and closer.
First of all, we prove that under the hypothesis of multiplicity one of the blow–up limits, if more than two triple junctions go to collide, then necessarily an entire region (the interior of a "loop" of the network) vanishes, which implies that the curvature is necessarily unbounded getting close to the singular time. Hence, if the curvature stays bounded it must happen that (locally) we are in the case of two triple junctions (only) going to collide along a vanishing curve, forming a 4–point in the limit. Vice versa, we are then able to show that in such a situation the curvature remains bounded. As a consequence, we conclude that the curvature is uniformly bounded along the flow if and only if no region is collapsing and that in such case only local vanishing of single curves can happen, with a formation of a 4–point in the limit. This is clearly particularly relevant if the evolving network is a *tree*, that is, regions are not present at all. More in detail, we first show that in such case, as t goes to the maximal time T, the networks $\mathbb{S}_t$ converge in C^1–norm (up to reparametrization) to a unique limit set $\mathbb{S}_T$ which is a *degenerate* (collapsed) regular network (see Definition 8.1), that is, a smooth network possibly with multi–points of order higher than three and some collapsed parts "hidden" in its vertices. Then, we show that $\mathbb{S}_T$ can have only 3–points with angles of 120 degrees or 4–points with angles of 120/60 degrees, like in the left side of Figure 10.1.
In the other situation, when the curvature is not bounded and a region collapses (Section 10.3), we are able to obtain a weaker conclusion. Assuming the uniqueness of the blow–up limit along any sequence of rescalings (which can be instead proved in the above case), we can show that, as $t \to T$, the network $\mathbb{S}_t$ converges to some degenerate (see above) regular network, whose "non–collapsed" part $\mathbb{S}_T$ is a C^1, possibly non–regular, network which is smooth outside its multi–points and whose curvature is of order $o(1/r)$, where r is the distance from its non–regular multi–points.
In several steps of the previous analysis the assumption of multiplicity one of the blow–up limits is fundamental, we actually conjecture (Conjecture 10.1) that it holds in general, but up to now we can prove it only in some special cases. Indeed, in Chapter 14 we discuss a scaling invariant, geometric quantity associated with a network, first proposed in [49] (see also [52]) and later extended in [15, 82, 89], consisting in a sort of "embeddedness measure" which is positive when no self–intersections are present. By a monotonicity argument, we show that this quantity is uniformly positively bounded below along the flow, under the assumption that the number of 3–points of the network is at most two. As a consequence, in such case every possible C^1_{loc}–limit of rescalings of the networks of the flow is an embedded network with multiplicity one. We underline that it is not clear to us how to obtain a similar conclusion for a general network with several triple junctions, since the analogous quantity, if there are more than two 3–points, does not satisfy a monotonicity property.

In Chapter 11 we state a short–time existence result for possibly non–regular initial networks (that is, with multi–points of order greater than 3 and/or non–regular 3–points), giving a flow that is immediately regular and smooth for every positive time. This result, which clearly also provides a "restarting theorem", was worked out independently in [58] by T. Ilmanen, A. Neves and the fourth author (Theorem 11.9) and in [70] by J. Lira, R. Mazzeo, M. Saez and the third author (Theorem 11.26), here we only give an outline of the arguments in the proofs (which are quite technical). The idea in Theorem 11.9 is to locally desingularize the multi–points and the non–regular 3–points via regular self–similarly expanding solutions. The argument hinges on a new monotonicity formula, which shows that such expanding solutions are dynamically stable, using the fact that the evolution of curves and networks in the plane are special cases of the Lagrangian mean curvature flow (these ideas have already been exploited by A. Neves in the papers [84–86]). Theorem 11.26 relies instead on blow–up arguments from geometric micro–local analysis. In this case, the same regular self–similarly expanding solutions naturally arise from the underlying geometric structure of the problem.
In Chapter 12 it is explained how to combine Theorem 11.9 with the previous analysis of the singularities in order to continue the flow after a singular time. Then, we analyze the preserved geometric quantities and the possible changes in the topology of a network in passing through a singularity. This is applied in Chapter 13 to study the long-time behavior of the flow, indeed, the restarting procedure allows us to define an "extended" curvature flow with singularities at an increasing sequence of times. An important open question is whether the maximal time interval of existence of such flow is finite or not, where the main problem is the possible "accumulation" of the singular times (if they are not finite, which actually we do not know). We mention that in the special case of symmetric networks with only two triple junctions, it can be shown that the set of singular times is necessarily finite, see [88]. Clearly, if such "extended" flow can be defined for every time (as the Brakke flow obtained by L. Kim and Y. Tonegawa in [62]), we ask ourselves if the network converges, as $t \to +\infty$, to a stationary network for the length functional (a Steiner network). In Proposition 13.6 we prove the convergence *up to a subsequence* of the family of the evolving networks to a possibly degenerate one (some curves could disappear in the limit), as $t \to +\infty$. If we then assume that such limit network is not degenerate, with the help of *Łojasiewicz–Simon gradient inequality*, we are actually able to prove the full convergence of the flow, in Theorem 13.11. We finally conclude Chapter 13 presenting a stability result: if a network is sufficiently close in $W^{2,2}$–norm to a regular network $\mathbb{S}_*$ composed of straight segments only, its motion by curvature exists for all times and smoothly converges to a regular network still composed of straight segments and with the same length of $\mathbb{S}_*$.

Up to now, the study of the behavior of the flow at the first singularity (and immediately after) is essentially complete when the network has at most two triple junctions, see [76, 80, 82, 89], holding in this very special case the above mentioned *multiplicity one conjecture*, as it is shown in Chapter 14. In Chapter 15 we will describe, up to the best of our knowledge, the global evolution of such "simple" networks, which are actually interesting since most of the relevant phenomena of the general case are already present. In particular, we will see that the evolution of a tree–like network with only one 3–point and three fixed end–points (called *triod*) is smooth and asymptotically converges to a Steiner network, if the lengths of the three curves stay uniformly bounded away from zero.

The last section of the paper is devoted to collecting and presenting the main open problems. Moreover, by courtesy of T. Ilmanen, we include an appendix with pictures and computations of several examples of regular shrinkers, due to him and J. Hättenschweiler.

We conclude this introduction by mentioning that there are several interesting variants and generalizations of the problem of the motion by curvature of networks whose study is only at the beginning. For instance, one can consider the anisotropic version of the flow, as in [13, 45, 64] and/or take into account the mismatch of the orientation of the grain in the model [32, 33, 59].

The analogous problem in higher dimensions (and codimensions) is still widely open. Besides the papers [62, 98], where a global weak solution in the Brakke sense is constructed, the short–time existence of a smooth and regular solution in three dimensions has been established in [28] in some special cases and in [98, Section 7] for the motion of a network in $\mathbb{R}^n$ with only triple junctions. In these cases, the analysis of singularities and the subsequent possible restarting procedure are still open problems.

We also mention the works [37, 38] where a graph evolving by mean curvature and meeting a horizontal hyperplane with a fixed angle of 60 degrees is studied. By considering the union of such graph with its reflection through the hyperplane, one gets an evolving symmetric *lens–shaped* domain. We remark that in this particular case, the analysis is simpler since the maximum principle can be applied.

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