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DYNAMICS OF THE DOMINANT HAMILTONIAN

by Vadim Kaloshin & Ke Zhang

Abstract. — It is well known that instabilities of nearly integrable Hamiltonian systems occur around resonances. Dynamics near resonances of these systems is well approximated by the associated averaged system, called *slow system*. Each resonance is defined by a basis (a collection of integer vectors). We introduce a class of resonances whose basis can be divided into two well separated groups and call them *dominant*. We prove that the associated slow system can be well approximated by a subsystem given by one of the groups, both in the sense of the vector field and weak KAM theory. As a corollary, we obtain perturbation results on normally hyperbolic invariant cylinders, and the Aubry/Mañe sets. This has applications in Arnold diffusion in arbitrary degrees of freedom.

Résumé (*Dynamique de l'hamiltonien dominant*). — Il est bien connu que les instabilités des systèmes hamiltoniens presque intégrables interviennent au voisinage des résonances. La dynamique de ces systèmes près des résonances est bien approchée par les systèmes moyennés associés, appelés *systèmes lents*. Chaque résonance est définie par une base (une collection de vecteurs entiers). Nous introduisons une classe de résonances dont la base peut être divisée en deux groupes bien distincts, que nous appelons *dominantes*. Nous prouvons que le système lent associé peut être bien approché par un sous-système donné par l'un de ces deux groupes, à la fois comme champ de vecteurs et au sens de la théorie KAM faible. Comme corollaire, nous obtenons des résultats perturbatifs sur des cylindres invariants normalement hyperboliques, et sur

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les ensembles d'Aubry/Mañé. Cela a des applications en diffusion d'Arnold pour un nombre arbitraire de degrés de liberté.

1. Introduction

Consider a nearly integrable system with $n\frac{1}{2}$ degrees of freedom

$$H_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon H_1(\theta, p, t), \quad \theta \in \mathbb{T}^n, p \in \mathbb{R}^n, t \in \mathbb{T}. \tag{1.1}$$

We will restrict to the case where the integrable part H_0 is strictly convex, more precisely, we assume that there is $D > 1$ such that

$$D^{-1}\,\mathrm{Id} \leqslant \partial^2_{pp} H_0(p) \leqslant D\,\mathrm{Id}$$

as quadratic forms, where Id denotes the identity matrix.

The main motivation behind this work is the question of Arnold diffusion, that is, topological instability for the system H_ε. Arnold provided the first example in [3], and asks ([1, 2, 4]) whether topological instability is "typical" in nearly integrable systems with $n \geqslant 2$ (the system is stable when $n = 1$, due to low dimensionality).

It is well known that the instabilities of nearly integrable systems occurs along resonances. Given an integer vector $k = (\bar{k}, k^0) \in \mathbb{Z}^n \times \mathbb{Z}$ with $\bar{k} \neq 0$, we define the resonant submanifold to be $\Gamma_k = \{p \in \mathbb{R}^n : k \cdot (\omega(p), 1) = 0\}$, where $\omega(p) = \partial_p H_0(p)$. More generally, we consider a subgroup Λ of $\mathbb{Z}^{n+1}$ which does not contain vectors of the type $(0, \dots, 0, k^0)$, called a *resonance lattice*. The *rank* of Λ is the dimension of the real subspace containing it. Then for a rank d resonance lattice Λ, we define

$$\Gamma_\Lambda = \bigcap \{\Gamma_k : k \in \Lambda\} = \bigcap_{i=1}^{d} \Gamma_{k_i},$$

where $\{k_1, \dots, k_d\}$ is any linear independent set in Λ. We call such Γ_Λ a *d-resonance submanifold* (d-resonance for short), which is a co-dimension d submanifold of $\mathbb{R}^n$, and in particular, an n-resonant submanifold is a single point. We say that Λ is *irreducible* if it is not contained in any lattices of the same rank, or equivalently, $\mathrm{span}_{\mathbb{R}}\Lambda \cap \mathbb{Z}^{n+1} = \Lambda$.

We now consider the diffusion that occurs along a connected net Γ of $(n-1)$-resonances, which are curves in $\mathbb{R}^n$. The main difficulty in proving Arnold diffusion is in crossing the maximal (n-)resonances, which are intersections of Γ with a transveral 1-resonance manifold $\Gamma_{k'}$. A similar question is whether one can "switch" at the intersection of two resonant curves (see Figure 1.1).

For an n-resonance $\{p_0\} = \Gamma_\Lambda$, we assume that Λ is irreducible, and $\mathcal{B} = [k_1, \dots, k_n]$ is a basis over $\mathbb{Z}$. The averaging theory of H_ϵ near p_0 reduces to

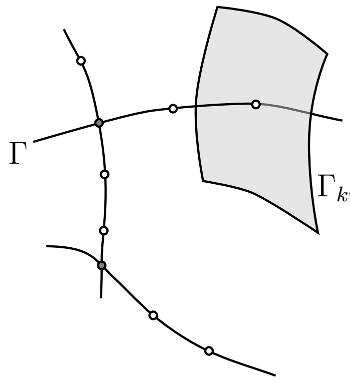

FIGURE 1.1. Diffusion path and essential resonances in $n = 3$. The hollow dots requires crossing, while the gray dots requires switching

the study of a particular *slow system* defined on $\mathbb{T}^n \times \mathbb{R}^n$, denoted $H^s_{p_0,\mathcal{B}}$. More precisely, in an $O(\sqrt{\varepsilon})$-neighborhood of p_0, the system H_ε admits the normal form (see [16], Appendix B)

$$H^s_{p_0,\mathcal{B}}(\varphi, I) + \sqrt{\varepsilon}P(\varphi, I, \tau), \quad \varphi \in \mathbb{T}^n, I \in \mathbb{T}^n, \tau \in \sqrt{\varepsilon}\mathbb{T},$$

where

$$\varphi_i = k_i \cdot (\theta, t),\ 1 \leqslant i \leqslant n, \quad (p - p_0)/\sqrt{\epsilon} = \bar{k}_1 I_1 + \cdots \bar{k}_n I_n.$$

Therefore, H_ϵ is conjugate to a fast periodic perturbation to $H^s_{p_0,\mathcal{B}}$. Note that our definition depend on the choice of basis $\mathcal{B}$. A basis free definition requires using a non-standard torus $\mathbb{T}^{n+1}/\omega(p_0)\mathbb{R}$ as the configuration space, and in this paper we choose to avoid this setting and fix a basis. Such averaged systems were studied in [24].

When $n = 2$, the slow system is a 2 degree of freedom mechanical system, the structure of its (minimal) orbits is well understood. This fact underlies the results on Arnold diffusion in two and half degrees of freedom (see [23], [24], [25], [10], [16], [14], [17], [20], [21]). This is no longer the case when $n > 2$, which is a serious obstacle to proving Arnold diffusion in higher degrees of freedom. In [15] it is proposed that we can sidestep this difficulty by using *dimension reduction*: using existence of normally hyperbolic invariant cylinders (NHICs) to restrict the system to a lower dimensional manifold. This approach only works when the slow system has a particular *dominant structure*, which is the topic of this paper.

In order to make this idea specific it is convenient to define the slow system for any p_0 and any d-resonance $d \leqslant n$. For $p_0 \in \mathbb{R}^n$, an irreducible rank d

resonance lattice Λ, and its basis $\mathcal{B} = [k_1, \dots, k_d]$, the slow system is

$$H^s_{p_0,\mathcal{B}}(\varphi, I) = K_{p_0,\mathcal{B}}(I) - U_{p_0,\mathcal{B}}(\varphi), \quad \varphi \in \mathbb{T}^d, I \in \mathbb{T}^d. \tag{1.2}$$

Suppose the Fourier expansion of H_1 is $\sum_{k\in\mathbb{Z}^{n+1}} h_k(p)e^{2\pi i k\cdot(\theta,t)}$, then

$$K_{p_0,\mathcal{B}}(I) = \frac{1}{2}\partial^2_{pp}H_0(p_0)(I_1\bar{k}_1 + \cdots + I_d\bar{k}_d)\cdot(I_1\bar{k}_1 + \cdots + I_d\bar{k}_d), \tag{1.3}$$

$$U_{p_0,\mathcal{B}}(\varphi_1, \dots, \varphi_d) = -\sum_{l\in\mathbb{Z}^d} h_{l_1k_1+\cdots l_dk_d}(p_0)\, e^{2\pi i(l_1\varphi_1+\cdots+l_d\varphi_d)}. \tag{1.4}$$

The system $H^s_{p_0,\mathcal{B}}$ is only dynamically meaningful when $p_0 \in \Gamma_\Lambda$. However, the more general set up allows us to embed the meaningful slow systems into a nice space.

In the sequel we fix a rank $m < n$ lattice, called the *strong lattice*, and its basis $\mathcal{B} = [k_1, \dots, k_m]$. We say an irreducible lattice $\Lambda \supset \Lambda^{\mathrm{st}}$ of rank d is *dominated* by Λ^{st} if

$$M(\Lambda|\Lambda^{\mathrm{st}}) := \min_{k\in\Lambda\setminus\Lambda^{\mathrm{st}}} |k| \gg \max_{k\in\mathcal{B}^{\mathrm{st}}} |k|, \tag{1.5}$$

where $|k| = \sup_i |k_i|$ is the sup-norm. Given the relation $\Lambda^{\mathrm{st}} \subset \Lambda$, we extend the basis $[k_1, \dots, k_m]$ of Λ^{st} to a basis $\mathcal{B} = [k_1, \dots, k_d]$ of Λ, such a basis is called *adapted.* Naturally, as $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$, we have $|k_{m+1}|, \dots, |k_d| \to \infty$ for any adapted basis.

While we have fixed the basis $\mathcal{B}^{\mathrm{st}}$ of Λ^{st}, the system $H_{p_0,\mathcal{B}}$ strongly depends on the choice of the adapted basis. To get a meaningful result, we only consider particular bases that we call *κ-ordered.* Roughly speaking, given $\kappa > 1$, a basis $[k_1, \dots, k_d]$ is κ-ordered if k_i is, up to a factor of order κ, the vector of smallest norm in the set $\Lambda\setminus\mathrm{span}_{\mathbb{Z}}\{k_1, \dots, k_{i-1}\}$. The precise definition of this basis is given in Section 2.2. We will show that there exists κ depending only on $\mathcal{B}^{\mathrm{st}}$, such that any $\Lambda \supsetneq \Lambda^{\mathrm{st}}$ admits a κ-ordered basis.

After an ordered basis is chosen, we have two systems $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ and $H^s_{p_0,\mathcal{B}}$, which we call the *strong system* and *slow system* respectively. When the lattices have a dominant structure (see (1.5)), the slow system $H^s_{p_0,\mathcal{B}}$ inherits considerable amount of information from the strong system. Indeed, let us denote

$$\begin{aligned} H^s_{p_0,\mathcal{B}^{\mathrm{st}}} &= K^{\mathrm{st}}(I_1, \dots, I_m) - U^{\mathrm{st}}(\varphi_1, \dots, \varphi_m), \\ H^s_{p_0,\mathcal{B}} &= K(I_1, \dots, I_d) - U(\varphi_1, \dots, \varphi_d), \end{aligned}$$

under (1.5) and we will show that

$$K^{\mathrm{st}}(I_1, \dots, I_m) = K(I_1, \dots, I_m, 0, \dots, 0), \quad \|U - U^{\mathrm{st}}\|_{C^2} \ll 1, \tag{1.6}$$

which indicates $H^s_{p_0,\mathcal{B}}$ can be approximated by an *extension* of $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$. The variables φ_i, I_i, $1 \leqslant i \leqslant m$ are called the *strong variables*, while φ_i, I_i, $m+1 \leqslant i \leqslant d$ are called the *weak variables.*

Recall that for each convex Hamiltonian H, we can associate a Lagrangian $L = L_H$, and the Euler-Lagrange flow is conjugate to the Hamiltonian flow. Denote by $X^{\mathrm{st}}_{\mathrm{Lag}}$ and X^s_{Lag} the Euler-Lagrange vector fields associated to the Hamiltonians $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ and $H^s_{p_0,\mathcal{B}}$. Since the system for $X^{\mathrm{st}}_{\mathrm{Lag}}$ is only defined for the strong variables (φ_i, v_i), $1 \leqslant i \leqslant m$, we define a *trivial extension* of $X^{\mathrm{st}}_{\mathrm{Lag}}$ by setting $\dot\varphi_i = \dot v_i = 0$, $m+1 \leqslant i \leqslant d$.

We show that after performing a coordinate change[(1)] and rescaling transformation in the weak variables, the transformed vector field X^s_{Lag} converges to that of $X^{\mathrm{st}}_{\mathrm{Lag}}$ in some sense. In particular, if $X^{\mathrm{st}}_{\mathrm{Lag}}$ admits a normally hyperbolic invariant cylinder (NHIC), so does X^s_{Lag}. In a separate direction, we also obtain a limit theorem on the weak KAM solutions by variational arguments. We now formulate our main results in loose language, leaving the precise version for the next section.

Main Result. — *Assume that $r > n + 2(d-m) + 4$. Given a fixed lattice Λ^{st} of rank m with a fixed basis $\mathcal{B}^{\mathrm{st}}$, there exist $\kappa > 1$ depending only on $\mathcal{B}^{\mathrm{st}}$, and the following hold. Each rank $d, m \leqslant d \leqslant n$ irreducible lattice $\Lambda \supset \Lambda^{\mathrm{st}}$ admits a κ-ordered basis, under which we have:*

1. *(Geometrical) As $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$, the projection of X^s_{Lag} to the strong variables (φ_i, v_i), $1 \leqslant i \leqslant m$ converges to $X^{\mathrm{st}}_{\mathrm{Lag}}$ uniformly. Moreover, by introducing a coordinate change and rescaling affecting only the weak variables (φ_i, v_i), $m+1 \leqslant i \leqslant d$, the transformed vector field of X^s_{Lag} converges to a trivial extension of the vector field of $X^{\mathrm{st}}_{\mathrm{Lag}}$. As a corollary, we obtain that if $X^{\mathrm{st}}_{\mathrm{Lag}}$ admits an NHIC, then so does X^s_{Lag} for sufficiently large $M(\Lambda|\Lambda^{\mathrm{st}})$.*
2. *(Variational) If, in addition, we have $r > n + 4(d-m) + 4$, then as $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$, the weak KAM solution of $H^s_{p_0,\mathcal{B}}$ (of properly chosen cohomology classes) converges uniformly to a trivial extension of a weak KAM solution of $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$, considered as functions on $\mathbb{R}^d$. We also obtain corollaries concerning the limits of Mañe, Aubry sets, rotation vector of minimal measure, and Peierl's barrier function. The precise definitions of these objects will be given later.*

The statement that $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ approximates $H^s_{p_0,\mathcal{B}}$ is related to the classical concept of partial averaging (see for example [5]). The statement $\min_{k\in\Lambda\setminus\Lambda^{\mathrm{st}}} |k| \gg \max_{k\in\mathcal{B}^{\mathrm{st}}} |k|$ says that the resonances in Λ^{st} is much *stronger* than the rest of the resonances in Λ. Partial averaging says that the weaker resonances contributes to smaller terms in a normal form.

However, our treatment of the partial averaging theory is quite different from the classical theory. By looking at the rescaling limit, we study the property

1. The coordinate change we perform is known in analytic mechanics as the Routhian coordinates, which is an half-Lagrangian, half-Hamiltonian setting, see (2.8).