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COHOMOLOGY CLASSES REPRESENTED BY MEASURED FOLIATIONS, AND MAHLER'S QUESTION FOR INTERVAL EXCHANGES

BY Yair MINSKY AND Barak WEISS

Abstract. – A translation structure on (S, Σ) gives rise to two transverse measured foliations $\mathcal{F}, \mathcal{G}$ on S with singularities in Σ, and by integration, to a pair of relative cohomology classes $[\mathcal{F}]$, $[\mathcal{G}] \in H^1(S, \Sigma; \mathbb{R})$. Given a measured foliation $\mathcal{F}$, we characterize the set of cohomology classes $\mathbf{b}$ for which there is a measured foliation $\mathcal{G}$ as above with $\mathbf{b} = [\mathcal{G}]$. This extends previous results of Thurston [19] and Sullivan [18].

We apply this to two problems: unique ergodicity of interval exchanges and flows on the moduli space of translation surfaces. For a fixed permutation $\sigma \in \mathcal{S}_d$, the space $\mathbb{R}^d_+$ parametrizes the interval exchanges on d intervals with permutation σ. We describe lines ℓ in $\mathbb{R}^d_+$ such that almost every point in ℓ is uniquely ergodic. We also show that for $\sigma(i) = d+1-i$, for almost every $s > 0$, the interval exchange transformation corresponding to σ and $(s, s^2, \ldots, s^d)$ is uniquely ergodic. As another application we show that when $k = |\Sigma| \geq 2$, the operation of "moving the singularities horizontally" is globally well-defined. We prove that there is a well-defined action of the group $B \ltimes \mathbb{R}^{k-1}$ on the set of translation surfaces of type (S, Σ) without horizontal saddle connections. Here $B \subset \mathrm{SL}(2, \mathbb{R})$ is the subgroup of upper triangular matrices.

Résumé. – Une structure de translation sur une surface marquée (S, Σ) donne lieu à deux feuilletages mesurés $\mathcal{F}$, $\mathcal{G}$ sur S à singularités dans Σ et, par intégration, à un couple de classes de cohomologie relative $[\mathcal{F}]$, $[\mathcal{G}] \in H^1(S, \Sigma; \mathbb{R})$. Étant donné un feuilletage mesuré $\mathcal{F}$, nous caractérisons l'ensemble des classes de cohomologie $\mathbf{b}$ pour lesquelles il existe un feuilletage mesuré $\mathcal{G}$ comme ci-dessus tel que $\mathbf{b} = [\mathcal{G}]$. Cela généralise des résultats antérieurs de Thurston [19] et Sullivan [18].

Nous appliquons ce résultat à deux problèmes : l'unique ergodicité des échanges d'intervalles et les flots sur l'espace des modules des surfaces de translation. Étant donnée une permutation $\sigma \in \mathcal{S}_d$, l'ensemble $\mathbb{R}^d_+$ paramètre les échanges d'intervalles sur d intervalles de permutation associée σ. Nous décrivons les droites ℓ de $\mathbb{R}^d_+$ dont presque tout point est uniquement ergodique. Nous démontrons aussi que si σ est donnée par $\sigma(i) = d + 1 - i$, pour presque tout $s > 0$, l'échange d'intervalles correspondant à σ et à $(s, s^2, \ldots, s^d)$ est uniquement ergodique. Une autre application est que lorsque $k = |\Sigma| \geq 2$, l'opération consistant à « déplacer horizontalement les singularités » est bien définie. En notant B le sous-groupe des matrices triangulaires supérieures de $\mathrm{SL}(2, \mathbb{R})$, nous prouvons qu'il y a une action bien définie du groupe $B \times \mathbb{R}^{k-1}$ sur l'ensemble des surfaces de translation de type (S, Σ) sans connexion horizontale.

1. Introduction

1.1. Motivating questions and nonsensical pictures

To introduce the problems discussed in this paper, consider some pictures. Suppose that $\mathbf{a} = (a_1, \ldots, a_d)$ is a vector with positive entries, $I = [0, \sum a_i)$ is an interval, σ is a permutation on d symbols, and $\mathcal{T} = \mathcal{T}_\sigma(\mathbf{a}) : I \to I$ is the interval exchange obtained by cutting up I into segments of lengths a_i and permuting them according to σ. A fruitful technique for studying the dynamical properties of $\mathcal{T}$ is to consider it as the return map to a transverse segment along the vertical foliation in a translation surface, i.e., a union of polygons with edges glued pairwise by translations. See Figure 1.1 for an example with one polygon; note that the interval exchange determines the horizontal coordinates of vertices, but there are many possible choices of the vertical coordinates.

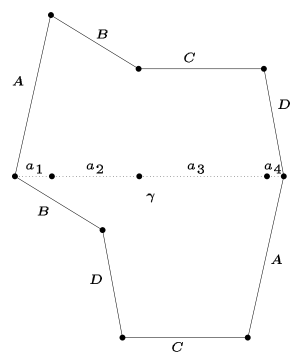

FIGURE 1.1. Masur's construction ([9], according to Masur this was suggested by the referee of [9]): an interval exchange embedded in a one-polygon translation surface.

Given a translation surface q with a transversal, one may deform it by applying the horocycle flow, i.e., deforming the polygon with the linear map

$$h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \tag{1}$$

The return map to a transversal in $h_s q$ depends on s, so we get a one-parameter family $\mathcal{T}_s$ of interval exchange transformations (Figure 1.2). For sufficiently small s, one has $\mathcal{T}_s = \mathcal{T}_\sigma(\mathbf{a}(s))$, where $\mathbf{a}(s) = \mathbf{a} + s\mathbf{b}$ is a line segment, whose derivative $\mathbf{b} = (b_1, \ldots, b_d)$ is determined by the *heights* of the vertices of the polygon. We will consider an inverse problem: given a line segment $\mathbf{a}(s) = \mathbf{a} + s\mathbf{b}$, does there exist a translation surface q such that for all sufficiently small s, $\mathcal{T}_\sigma(\mathbf{a}(s))$ is the return map along vertical leaves to a transverse segment in $h_s q$? Attempting to interpret this question with pictures, we see that some choices of $\mathbf{b}$ lead to a translation surface while others lead to nonsensical pictures—see Figure 1.3. The solution to this problem is given by Theorem 5.3.

Now consider a translation surface q with two singularities. We may consider the operation of moving one singularity horizontally with respect to the other. That is, at time s, the line segments joining one singularity to the other are made longer by s, while line segments

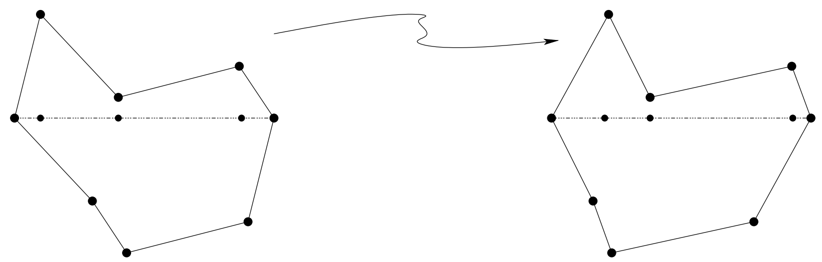

FIGURE 1.2. The horocycle action gives a linear one parameter family of interval exchanges

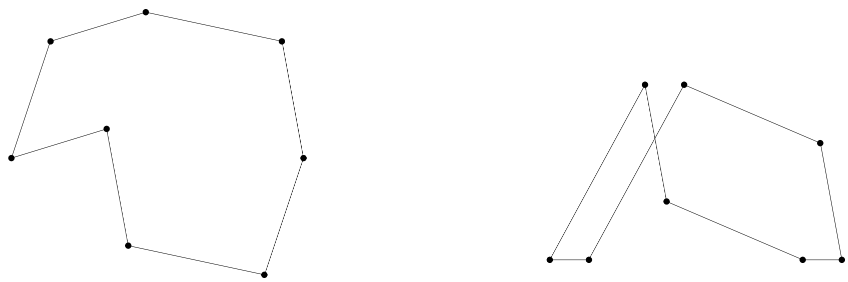

FIGURE 1.3. The choice $\mathbf{b} = (2, 1, -1, -2)$ (left) gives a translation surface but what about $\mathbf{b} = (0, 3, -1, -2)$?

joining a singularity to itself are unchanged. For small values of s, one obtains a new translation surface q_s by examining the picture. But for large values of s, some of the segments in the figure cross each other and it is not clear whether the operation defined above gives rise to a well-defined surface. Our Theorem 11.2 shows that the operation of moving the zeroes is well-defined for all values of s, provided one rules out the obvious obstruction that two singularities connected by a horizontal segment collide.

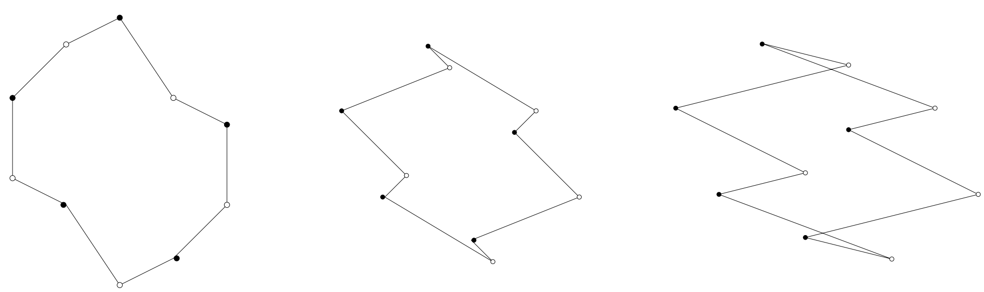

FIGURE 1.4. The singularity $\circ$ is moved to the right with respect to $\bullet$, and the picture becomes nonsensical.

1.2. Main geometrical result

Let S be a compact oriented surface of genus $g \geq 2$ and $\Sigma \subset S$ a finite subset. A *translation surface structure* on (S, Σ) is an atlas of charts into the plane, whose domains cover $S \setminus \Sigma$, and such that the transition maps are translations. Such structures arise naturally in complex analysis and in the study of interval exchange transformations and polygonal billiards and have been the subject of intensive research, see the recent surveys [12, 25].

Several geometric structures on the plane can be pulled back to $S \setminus \Sigma$ via the atlas, among them the foliations of the plane by horizontal and vertical lines. We call the resulting oriented foliations of $S \setminus \Sigma$ the *horizontal and vertical foliation* respectively. Each can be completed to a singular foliation on S, with a pronged *singularity* at each point of Σ. Label the points of Σ by $\xi_1, \ldots, \xi_k$ and fix natural numbers $r_1, \ldots, r_k$. We say that the translation surface is *of type* $\mathbf{r} = (r_1, \ldots, r_k)$ if the horizontal and vertical foliations have a $2(r_j+1)$-pronged singularity at each ξ_j.

By pulling back dy (resp. dx) from the plane, the horizontal (vertical) foliation arising from a translation surface structure is equipped with a *transverse measure*, i.e., a family of measures on each arc transverse to the foliation which is invariant under holonomy along leaves. We will call an oriented singular foliation on S, with singularities in Σ, which is equipped with a transverse measure a *measured foliation on* (S, Σ). We caution the reader that we deviate from the convention adopted in several papers on this subject, by considering the number and orders of singularities as part of the structure of a measured foliation; we call these the *type* of the foliation. In other words, we do not consider two measured foliations which differ by a Whitehead move to be the same.

Integrating the transverse measures gives rise to two well-defined cohomology classes in the relative cohomology group $H^1(S, \Sigma; \mathbb{R})$. That is we obtain a map

$$\mathrm{hol} : \{\text{translation surfaces on } (S, \Sigma)\} \to \left(H^1(S, \Sigma; \mathbb{R})\right)^2.$$

This map is a local homeomorphism and serves to give coordinate charts to the set of translation surfaces (see § 2.1 for more details), but it is not globally injective. For example, precomposing with a homeomorphism which acts trivially on homology may change a marked translation surface structure but does not change its image in under hol; see [14] for more examples. On the other hand it is not hard to see that the pair of horizontal and vertical measured foliations associated to a translation surface uniquely determine it, and hence the question arises of reconstructing the translation surface from just the cohomological data recorded by hol. Our main theorems give results in this direction.

To state them we define the set of *(relative) cycles carried by* $\mathcal{F}$, denoted $H_+^{\mathcal{F}}$, to be the image in $H_1(S, \Sigma; \mathbb{R})$ of all (possibly atomic) transverse measures on $\mathcal{F}$ (see § 2.5).

THEOREM 1.1. – *Suppose $\mathcal{F}$ is a measured foliation on (S, Σ), and $\mathbf{b} \in H^1(S, \Sigma; \mathbb{R})$. Then the following are equivalent:*

1. *There is a measured foliation $\mathcal{G}$ on (S, Σ), everywhere transverse to $\mathcal{F}$ and of the same type, such that $\mathcal{G}$ represents $\mathbf{b}$.*
2. *Possibly after replacing $\mathbf{b}$ with $-\mathbf{b}$, for any $\delta \in H_+^{\mathcal{F}}$, $\mathbf{b} \cdot \delta > 0$ ($\mathbf{b} \cdot \delta$ denotes the natural pairing of homology and cohomology).*

After proving Theorem 1.1 we learned from F. Bonahon that it has a long and interesting history. Similar result were proved by Thurston [19] in the context of train tracks and measured laminations, and by Sullivan [18] in a very general context involving foliations. Bonahon neglected to mention his own contribution [1]. Our result is a 'relative version' in that we