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GROMOV HYPERBOLICITY AND QUASIHYPERBOLIC GEODESICS

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ABSTRACT. – We characterize Gromov hyperbolicity of the quasihyperbolic metric space (Ω, k) by geometric properties of the Ahlfors regular length metric measure space (Ω, d, μ). The characterizing properties are called the Gehring-Hayman condition and the ball-separation condition.

RÉSUMÉ. – Nous caractérisons l'hyperbolicité au sens de Gromov de l'espace quasi-hyperbolique (Ω, k) par des propriétés géométriques (dites condition de Gehring-Hayman et condition de séparation des boules) de l'espace métrique mesuré Ahlfors-régulier (Ω, d, μ).

1. Introduction

Given a proper subdomain Ω of Euclidean space $\mathbb{R}^n$, $n \geq 2$, equipped with Euclidean distance, one defines the *quasihyperbolic metric* k in Ω as the path metric generated by the density

$$\rho(z) = \frac{1}{d(z)},$$

where $d(z) = \operatorname{dist}(z, \partial\Omega)$. Precisely, one sets

$$k(x, y) = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \rho(z)\, ds,$$

where the infimum is taken over all rectifiable curves γ_{xy} that join x and y in Ω and the integral is the usual line integral. Then Ω equipped with k is a geodesic metric space: there is a curve γ_{xy} whose length in the above sense equals $k(x, y)$. Let us denote by $[x, y]$ any such geodesic; these geodesics are not necessarily unique as can be easily seen, for example for $\Omega = \mathbb{R}^n \setminus \{0\}$. The quasihyperbolic metric k was introduced in [5] and [4] where the basic properties of it were established.

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If for all triples of geodesics $[x,y]$, $[y,z]$, $[z,x]$ in Ω every point in $[x,y]$ is within k-distance δ from $[y,z]\cup[z,x]$ then the space (Ω,k) is called *δ-hyperbolic.* Roughly speaking this means that *geodesic triangles* in Ω are *δ-thin.* Moreover, we say that (Ω,k) is *Gromov hyperbolic* if it is δ-hyperbolic for some δ. The following theorem from [1] that extends results from [2] gives a complete characterization of Gromov hyperbolicity of (Ω,k).

THEOREM 1.1. – *Let $\Omega\subset\mathbb{R}^n$ be a proper subdomain. Then (Ω,k) is Gromov hyperbolic if and only if Ω satisfies both a Gehring-Hayman condition and a ball separation condition.*

Above, the Gehring-Hayman condition means that there is a constant $C_{\mathrm{gh}}\geq 1$ such that for each pair of points x,y in Ω and for each quasihyperbolic geodesic $[x,y]$ it holds that

$$\mathrm{length}([x,y])\leq C_{\mathrm{gh}}\mathrm{length}(\gamma_{xy}),$$

where γ_{xy} is any other curve joining x to y in Ω. In other words, it says that quasihyperbolic geodesics are essentially the shortest curves in Ω.

The other condition, a ball separation condition, requires the existence of a constant $C_{\mathrm{bs}}\geq 1$ such that for each pair of points x and y, each quasihyperbolic geodesic $[x,y]$, every $z\in[x,y]$, and every curve γ_{xy} joining x to y it holds that

$$B(z,C_{\mathrm{bs}}d(z))\cap\gamma_{xy}\neq\varnothing.$$

Here the ball is taken with respect to the inner metric of Ω.

Notice that the three conditions in Theorem 1.1, Gromov hyperbolicity and the Gehring-Hayman and the ball separation conditions, are only based on metric concepts. It is then natural to ask for an extension of this characterization to an abstract metric setting. Such an extension was given in [1] that relies on an analytic assumption that essentially requires the space in question to support a suitable Poincaré inequality. This very same condition, expressed in terms of moduli of curve families [7], is already in force in [2].

The purpose of this paper is to show that Poincaré inequalities are not critical for geometric characterizations of Gromov hyperbolicity of a non-complete metric space equipped with the quasihyperbolic metric. Our main result reads as follows.

THEOREM 1.2. – *Let $Q>1$ and let (X,d,μ) be a Q-regular metric measure space with (X,d) a locally compact and annularly quasiconvex length space. Let Ω be a bounded and proper subdomain of X, and let d_Ω be the inner metric on Ω associated to d. Then (Ω,k) is Gromov hyperbolic if and only if (Ω,d_Ω) satisfies both a Gehring-Hayman condition and a ball separation condition.*

The main point in Theorem 1.2 is the necessity of the Gehring-Hayman and ball separation conditions; their sufficiency is already given in [1, Theorem 2.4 and Theorem 6.1].

Above, annular quasiconvexity means that there is a constant $\lambda\geq 1$ so that for any $x\in X$ and all $0<r'<r$ each pair of points y,z in $B(x,r)\setminus B(x,r')$ can be joined with a path γ_{yz} in $B(x,\lambda r)\setminus B(x,r'/\lambda)$ such that $\mathrm{length}(\gamma_{yz})\leq\lambda d(y,z)$, Q-regularity requires the existence of a constant C_{q} so that

$$r^Q/C_{\mathrm{q}}\leq\mu(B(x,r))\leq C_{\mathrm{q}}r^Q$$

for all $r > 0$ and all $x \in X$, and the other concepts are defined analogously to the Euclidean setting described in the beginning of our introduction. See Section 2 for the precise definitions. In fact, the assumptions of Theorem 1.2 can be somewhat relaxed, see Section 5.

This paper is organized as follows. Section 2 contains necessary definitions. In Section 3 we give preliminaries relating the quasihyperbolic metric and Whitney balls. Section 4 is devoted to the proof of our main technical estimate, and Section 5 contains the proof of our main result and some generalizations.

2. Definitions

Let (X, d) be a metric space. A *curve* is a continuous map $\gamma\colon [a, b] \to X$ from an interval $[a, b] \subset \mathbb{R}$ to X. We also denote the image set $\gamma([a, b])$ of γ by γ. The *length* $\ell_d(\gamma)$ of γ with respect to the metric d is defined as

$$\ell_d(\gamma) = \sup \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_m = b$ of the interval $[a, b]$. If $\ell_d(\gamma) < \infty$, then γ is said to be a *rectifiable curve.* When the parameter interval is open or half-open, we set

$$\ell_d(\gamma) = \sup \ell_d(\gamma|_{[c,d]}),$$

where the supremum is taken over all compact subintervals $[c, d]$.

When every pair of points in (X, d) can be joined with a rectifiable curve, the space (X, d) is called *rectifiably connected.* If $\ell_d(\gamma_{xy}) = d(x, y)$ for some curve γ_{xy} joining points $x, y \in X$, then γ_{xy} is said to be a *geodesic.* If every pair of points in (X, d) can be joined with a geodesic, then (X, d) is called a *geodesic space.* Moreover, a *geodesic ray* in X is an isometric image in (X, d) of the interval $[0, \infty)$. Furthermore, for a rectifiable curve γ we define the *arc length* $s\colon [a, b] \to [0, \infty)$ along γ by

$$s(t) = \ell_d(\gamma|_{[a,t]}).$$

Let (X, d) be a geodesic metric space and let $\delta \geq 0$. Denote by $[x, y]$ any geodesic joining two points x and y in X. If for all triples of geodesics $[x, y]$, $[y, z]$, $[z, x]$ in X every point in $[x, y]$ is within distance δ from $[y, z] \cup [z, x]$, the space (X, d) is called *δ-hyperbolic.* In other words, *geodesic triangles* in X are *δ-thin.* Moreover, we say that a space is *Gromov hyperbolic* if it is δ-hyperbolic for some δ. All Gromov hyperbolic spaces in this paper are assumed to be unbounded.

Next, let (X, d) be a locally compact, rectifiably connected and non-complete metric space, and denote by $\overline{X}_d$ its metric completion. Then the *boundary* $\partial_d X := \overline{X}_d \setminus X$ is nonempty. We write

$$d(z) := \operatorname{dist}_d(z, \partial_d X) = \inf\{d(z, x) \mid x \in \partial_d X\}$$

for $z \in X$.

Given a real number $D \geq 1$, a curve $\gamma\colon [a, b] \to X$ is called a *D-quasiconvex curve* if

$$\ell_d(\gamma) \leq D d(\gamma(a), \gamma(b)).$$

If γ also satisfies the *cigar condition*

$$\min\{\ell_d(\gamma|_{[a,t]}), \ell_d(\gamma|_{[t,b]})\} \le D d(\gamma(t))$$

for every $t \in [a, b]$, the curve is called a *D-uniform curve.* A metric space (X, d) is called a *D-quasiconvex space* or *D-uniform space* if every pair of points in it can be joined with a D-quasiconvex curve or a D-uniform curve respectively.

Let $\rho\colon X \to (0, \infty)$ be a continuous function. For each rectifiable curve $\gamma\colon [a, b] \to X$ we define the *ρ-length* $\ell_\rho(\gamma)$ of γ by

$$\ell_\rho(\gamma) = \int_\gamma \rho\, ds = \int_a^b \rho(\gamma(t))\, ds(t).$$

Because (X, d) is rectifiably connected, the density ρ determines a metric d_ρ, called the *ρ-metric,* defined by

$$d_\rho(x, y) = \inf_{\gamma_{xy}} \ell_\rho(\gamma_{xy}),$$

where the infimum is taken over all rectifiable curves γ_{xy} joining $x, y \in X$. If $\rho \equiv 1$, then $\ell_\rho(\gamma) = \ell_d(\gamma)$ is the length of the curve γ with respect to the metric d, and the metric $d_\rho = \ell_d$ is the *inner metric associated with d.* Generally, if the distance between every pair of points in the metric space is the infimum of the lengths of all curves joining the points, then the metric space is called a *length space.*

If we choose

$$\rho(z) = \frac{1}{d(z)},$$

we obtain the *quasihyperbolic metric* in X. In this special case, we denote the metric d_ρ by k and the quasihyperbolic length of the curve γ by $\ell_k(\gamma)$. Moreover, $[x, y]_k$ refers to a k-geodesic (i.e., quasihyperbolic geodesic) joining points x and y in X. Because we are dealing with many different metrics, the usual metric notations will have an additional subscript that refers to the metric in use. For ease of notation, terms which refer to the metric d_ρ will have an additional subscript ρ instead of d_ρ.

We say that (X, d) satisfies a *ball separation condition* if there is a constant $C_{\mathrm{bs}} \ge 1$ such that for each pair of points $x, y \in X$, for every k-geodesic $[x, y]_k \subset X$, for every $z \in [x, y]_k$, and for every curve γ_{xy} joining points x and y, it holds that

$$B_d(z, C_{\mathrm{bs}} d(z)) \cap \gamma_{xy} \neq \varnothing. \tag{BS}$$

Thus the condition says that the ball $B_d(z, C_{\mathrm{bs}} d(z))$ either includes at least one of the endpoints of the k-geodesic or it separates the endpoints. This condition was introduced in [2, §7]. We also say that (X, d) satisfies the *Gehring-Hayman condition* if there is a constant $C_{\mathrm{gh}} \ge 1$ such that for every k-geodesic $[x, y]_k$ it holds that

$$\ell_d([x, y]_k) \le C_{\mathrm{gh}} \ell_d(\gamma_{xy}), \tag{GH}$$

where γ_{xy} is any other curve joining x to y in X.

Following [1], we say that (X, d) is *minimally nice* if (X, d) is a locally compact, rectifiably connected and non-complete metric space, and the identity map from (X, d) to (X, ℓ_d) is continuous. If (X, d) is minimally nice, then the identity map from (X, d) to (X, k) is a homeomorphism, and (X, k) is complete (see [2, Theorem 2.8]); in particular, (X, k) is proper (i.e., closed balls are compact) and geodesic (recall the Hopf-Rinow theorem).