# ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Thomas ALAZARD & Jean-Marc DELORT

*Global solutions and asymptotic behavior
for two dimensional gravity water waves*

# Annales Scientifiques de l'École Normale Supérieure

# GLOBAL SOLUTIONS AND ASYMPTOTIC BEHAVIOR FOR TWO DIMENSIONAL GRAVITY WATER WAVES

BY Thomas ALAZARD AND Jean-Marc DELORT

ABSTRACT. – This paper is devoted to the proof of a global existence result for the water waves equation with smooth, small, and decaying at infinity Cauchy data. We obtain moreover an asymptotic description in physical coordinates of the solution, which shows that modified scattering holds.

The proof is based on a bootstrap argument involving $L^2$ and $L^\infty$ estimates. The $L^2$ bounds are proved in the companion paper [5] of this article. They rely on a normal forms paradifferential method allowing one to obtain energy estimates on the Eulerian formulation of the water waves equation. We give here the proof of the uniform bounds, interpreting the equation in a semi-classical way, and combining Klainerman vector fields with the description of the solution in terms of semi-classical Lagrangian distributions. This, together with the $L^2$ estimates of [5], allows us to deduce our main global existence result.

RÉSUMÉ. – Cet article est consacré à une preuve d'un résultat d'existence globale pour l'équation des ondes de gravité à données de Cauchy régulières, petites et décroissantes à l'infini. On obtient de plus une description asymptotique de la solution dans les coordonnées physiques, qui montre qu'il y a diffusion modifiée.

La démonstration est basée sur un argument inductif faisant intervenir des estimations *a priori* dans $L^2$ et $L^\infty$. Les bornes $L^2$ sont prouvées dans [5], texte complémentaire au présent article. Elles reposent sur une méthode de formes normales paradifférentielles permettant d'obtenir des estimations d'énergie sur la formulation eulérienne de l'équation des ondes de gravité. Nous donnons ici une démonstration des bornes uniformes, en interprétant l'équation de manière semi-classique, et en combinant la méthode des champs de vecteurs de Klainerman avec la description de la solution en termes de distributions lagrangiennes semi-classiques. Cela nous permet, compte tenu des estimations $L^2$ de [5], d'en déduire notre principal résultat d'existence globale.

## Introduction

### 1. Main result

Consider an homogeneous and incompressible fluid in a gravity field, occupying a time-dependent domain with a free surface. We assume that the motion is the same in every

vertical section and consider the two-dimensional motion in one such section. At time $t$, the fluid domain, denoted by $\Omega(t)$, is therefore a two-dimensional domain. We assume that its boundary is a free surface described by the equation $y = \eta(t, x)$, so that

$$\Omega(t) = \{ (x, y) \in \mathbb{R} \times \mathbb{R} \,;\, y < \eta(t, x) \}.$$

The velocity field is assumed to satisfy the incompressible Euler equations. Moreover, the fluid motion is assumed to have been generated from rest by conservative forces and is therefore irrotational in character. It follows that the velocity field $v \colon \Omega \to \mathbb{R}^2$ is given by $v = \nabla_{x,y} \phi$ for some velocity potential $\phi \colon \Omega \to \mathbb{R}$ satisfying

$$\text{(1)} \qquad \Delta_{x,y} \phi = 0, \quad \partial_t \phi + \frac{1}{2} \left| \nabla_{x,y} \phi \right|^2 + P + gy = 0,$$

where $g$ is the modulus of the acceleration of gravity ($g > 0$) and where $P$ is the pressure term. Hereafter, the units of length and time are chosen so that $g = 1$.

The problem is then given by two boundary conditions on the free surface:

$$\text{(2)} \qquad \begin{cases} \partial_t \eta = \sqrt{1 + (\partial_x \eta)^2}\, \partial_n \phi & \text{on } \partial\Omega, \\ P = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\partial_n$ is the outward normal derivative of $\Omega$, so that $\sqrt{1 + (\partial_x \eta)^2}\, \partial_n \phi = \partial_y \phi - (\partial_x \eta) \partial_x \phi$. The former condition expresses that the velocity of the free surface coincides with the one of the fluid particles. The latter condition is a balance of forces across the free surface.

Following Zakharov [69] and Craig and Sulem [27], we work with the trace of $\phi$ at the free boundary

$$\psi(t, x) := \phi(t, x, \eta(t, x)).$$

To form a system of two evolution equations for $\eta$ and $\psi$, one introduces the Dirichlet-Neumann operator $G(\eta)$ that relates $\psi$ to the normal derivative $\partial_n \phi$ of the potential by

$$(G(\eta)\psi)(t, x) = \sqrt{1 + (\partial_x \eta)^2}\, \partial_n \phi|_{y=\eta(t,x)}.$$

(This definition is made precise in the first section of the companion paper [5]. See Proposition 1.2 below). Then $(\eta, \psi)$ solves (see [27]) the so-called Craig-Sulem-Zakharov system

$$\text{(3)} \qquad \begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + \eta + \dfrac{1}{2}(\partial_x \psi)^2 - \dfrac{1}{2(1 + (\partial_x \eta)^2)} \big( G(\eta)\psi + (\partial_x \eta)(\partial_x \psi) \big)^2 = 0. \end{cases}$$

In [3], it is proved that if $(\eta, \psi)$ is a classical solution of (3), such that $(\eta, \psi)$ belongs to $C^0([0, T]; H^s(\mathbb{R}))$ for some $T > 0$ and $s > 3/2$, then one can define a velocity potential $\phi$ and a pressure $P$ satisfying (1) and (2). Thus it is sufficient to solve the Craig-Sulem-Zakharov formulation of the water waves equations.

Our main result is stated in full generality in the first section of this paper. A weaker statement is the following:

MAIN RESULT. – *For small enough initial data of size $\varepsilon \ll 1$, sufficiently decaying at infinity, the Cauchy problem for (3) is globally in time well-posed. Moreover, $u = |D_x|^{\frac{1}{2}} \psi + i\eta$*

*admits the following asymptotic expansion as t goes to $+\infty$: There is a continuous function $\underline{\alpha}\colon \mathbb{R} \to \mathbb{C}$, depending of $\varepsilon$ but bounded uniformly in $\varepsilon$, such that*

$$u(t,x) = \frac{\varepsilon}{\sqrt{t}}\underline{\alpha}\left(\frac{x}{t}\right)\exp\left(\frac{it}{4|x/t|} + \frac{i\varepsilon^2}{64}\frac{|\underline{\alpha}(x/t)|^2}{|x/t|^5}\log(t)\right) + \varepsilon t^{-\frac{1}{2}-\kappa}\rho(t,x)$$

*where $\kappa$ is some positive number and $\rho$ is a function uniformly bounded for $t \geq 1$, $\varepsilon \in \left]0,\varepsilon_0\right]$.*

As an example of small enough initial data sufficiently decaying at infinity, consider

(4)
$$\eta|_{t=1} = \varepsilon\eta_0, \quad \psi|_{t=1} = \varepsilon\psi_0,$$

with $\eta_0, \psi_0$ in $C_0^\infty(\mathbb{R})$. Then there exists a unique solution $(\eta, \psi)$ in $C^\infty([1, +\infty[; H^\infty(\mathbb{R}))$ of (3). In fact, in Theorem 1.4 we allow $\psi$ to be merely in some homogeneous Sobolev space.

The strategy of the proof will be explained in the following sections of this introduction. We discuss at the end of this paragraph some related previous works.

For the equations obtained by neglecting the nonlinear terms, the computation of the asymptotic behavior of the solutions was performed by Cauchy [17] who computed the phase of oscillations. The reader is referred to [31] and [30] for many historical comments on Cauchy's memoir.

Many results have been obtained in the study of the Cauchy problem for the water waves equations, starting from the pioneering work of Nalimov [55] who proved that the Cauchy problem is well-posed locally in time, in the framework of Sobolev spaces, under an additional smallness assumption on the data. We also refer the reader to Shinbrot [60], Yoshihara [68] and Craig [23]. Without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Wu for the case without surface tension (see [64, 65]) and by Beyer-Günther in [11] in the case with surface tension. Several extensions of their results have been obtained and we refer the reader to Córdoba, Córdoba and Gancedo [20], Coutand-Shkoller [21], Lannes [47, 49, 50], Linblad [51], Masmoudi-Rousset [53] and Shatah-Zeng [58, 59] for recent results on the Cauchy problem for the gravity water waves equation.

Our proof of global existence is based on the analysis of the Eulerian formulation of the water waves equations by means of microlocal analysis. In particular, the energy estimates discussed in [5] are influenced by the papers by Lannes [47] and Iooss-Plotnikov [44] and follow the paradifferential analysis introduced in [6] and further developed in [1, 4].

It is worth recalling that the only known coercive quantity for (3) is the Hamiltonian, which reads (see [69, 27])

(5)
$$\mathcal{H} = \frac{1}{2}\int \eta^2\,dx + \frac{1}{2}\int \psi G(\eta)\psi\,dx.$$

We refer to the paper by Benjamin and Olver [10] for considerations on the conservation laws of the water waves equations. One can compare the Hamiltonian with the critical threshold given by the scaling invariance of the equations. Recall (see [10, 18]) that if $(\eta, \psi)$ solves (3), then the functions $(\eta_\lambda, \psi_\lambda)$ defined by

(6)
$$\eta_\lambda(t,x) = \lambda^{-2}\eta\left(\lambda t, \lambda^2 x\right), \quad \psi_\lambda(t,x) = \lambda^{-3}\psi\left(\lambda t, \lambda^2 x\right) \qquad (\lambda > 0)$$

are also solutions of (3). In particular, one notices that the critical space for the scaling corresponds to $\eta_0$ in $\dot{H}^{3/2}(\mathbb{R})$. Since the Hamiltonian (5) only controls the $L^2(\mathbb{R})$-norm of $\eta$,