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P-ALCOVES AND NONEMPTINESS OF AFFINE DELIGNE-LUSZTIG VARIETIES

BY ULRICH GÖRTZ, XUHUA HE AND SIAN NIE

ABSTRACT. – We study affine Deligne-Lusztig varieties in the affine flag manifold of an algebraic group, and in particular the question, which affine Deligne-Lusztig varieties are non-empty. Under mild assumptions on the group, we provide a complete answer to this question in terms of the underlying affine root system. In particular, this proves the corresponding conjecture for split groups stated in [3]. The question of non-emptiness of affine Deligne-Lusztig varieties is closely related to the relationship between certain natural stratifications of moduli spaces of abelian varieties in positive characteristic.

RÉSUMÉ. – Nous étudions les variétés de Deligne-Lusztig affines dans la variété de drapeaux affine d'un groupe algébrique, et en particulier la question de savoir quelles variétés de Deligne-Lusztig affines sont non vides. À quelques restrictions près, nous donnons une réponse complète à cette question en termes de système de racines affine sous-jacent. Pour le cas des groupes déployés, cela résout en particulier la conjecture énoncée dans [3]. Ces propriétés sur les variétés de Deligne-Lusztig affines reflètent les relations entre certaines stratifications naturelles d'espaces de modules des variétés abéliennes en caractéristique positive.

1. Introduction

1.1. – Affine Deligne-Lusztig varieties (see below for the definition) are the analogues of Deligne-Lusztig varieties in the context of an affine root system, and hence are natural objects which deserve to be studied in their own interest. Furthermore, results about them have direct applications to certain questions in arithmetic geometry, specifically to moduli spaces of p-divisible groups and reductions of Shimura varieties. More concretely, if $\mathcal{M}$ is a Rapoport-Zink space, then $\mathcal{M}(\Bbbk)$ can be identified by Dieudonné theory with a (mixed-characteristic) affine Deligne-Lusztig variety. In this case, the formal scheme $\mathcal{M}$ provides a scheme structure. See [2, 5.10] for further information on this connection.

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1.2. – Let $\mathbb{F}_q$ be the finite field with q elements. Let $\Bbbk$ be an algebraic closure of $\mathbb{F}_q$. Consider one of the following two cases:

– Mixed characteristic case. Let $\mathbb{F}/\mathbb{Q}_p$ be a finite field extension with residue class field $\mathbb{F}_q$, and let $\mathbb{L}$ be the completion of the maximal unramified extension of $\mathbb{F}$. Denote by ε a uniformizer of $\mathbb{F}$.
– Equal characteristic case. Let $\mathbb{F} = \mathbb{F}_q((\epsilon))$, the field of Laurent series over $\mathbb{F}_q$, and $\mathbb{L} := \Bbbk((\epsilon))$, the field of Laurent series over $\Bbbk$. As in the previous case, $\mathbb{L}$ is the completion of the maximal unramified extension of $\mathbb{F}$.

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$ which splits over a tamely ramified extension of $\mathbb{F}$. Let σ be the Frobenius automorphism of $\mathbb{L}/\mathbb{F}$. We also denote the induced automorphism on $\mathbf{G}(\mathbb{L})$ by σ.

We fix a σ-invariant Iwahori subgroup $\mathbf{I} \subset \mathbf{G}(\mathbb{L})$. In the equal characteristic case we can view $\mathbf{G}(\mathbb{L})/\mathbf{I}$ as the $\Bbbk$-points of an ind-projective ind-scheme Flag over $\Bbbk$, the affine flag variety for $\mathbf{G}$, see [10]. The $\mathbf{I}$-double cosets in $\mathbf{G}(\mathbb{L})$ are parameterized by the extended affine Weyl group $\tilde{W}$. The automorphism on $\tilde{W}$ induced by σ is denoted by $\delta\colon \tilde{W} \to \tilde{W}$. Furthermore we denote by $\tilde{S} \subseteq \tilde{W}$ the set of simple affine reflections.

Following Rapoport [12], we define:

Definition 1.2.1. – *Let $x \in \tilde{W}$, and $b \in \mathbf{G}(\mathbb{L})$. The affine Deligne-Lusztig variety attached to x and b is the subset*

$$X_x(b) = \{g\mathbf{I} \in \mathbf{G}(\mathbb{L})/\mathbf{I};\ g^{-1}b\sigma(g) \in \mathbf{I}x\mathbf{I}\}.$$

In the equal characteristic case, it is not hard to see that there exists a unique locally closed $X_x(b) \subset$ Flag whose set of $\Bbbk$-valued points is the subset $X_x(b) \subseteq \mathbf{G}(\mathbb{L})/\mathbf{I}$ defined above. Moreover, $X_x(b)$ is a finite-dimensional $\Bbbk$-scheme, locally of finite type over $\Bbbk$ (but not in general of finite type: depending on b, $X_x(b)$ may have infinitely many irreducible components). In the mixed characteristic case, the term “variety” is not really justified. More precisely one should speak about affine Deligne-Lusztig sets.

As experience and partial results show, many basic properties of affine Deligne-Lusztig varieties such as non-emptiness and dimension depend only on the underlying combinatorial structure of the (affine) root system, and therefore coincide in the mixed characteristic and equal characteristic cases.

For the remainder of the introduction, we fix a basic element $b \in \mathbf{G}(\mathbb{L})$, i.e., an element whose Newton vector is central, or equivalently, whose σ-conjugacy class can be represented by a length zero element of the extended affine Weyl group. See [8] or [13] for more details.

So far, the main questions that have been studied are

1. For which x is $X_x(b) \neq \varnothing$?
2. If $X_x(b) \neq \varnothing$ and $X_x(b)$ carries a scheme structure, what is $\dim X_x(b)$?

Until recently, most of the results have been established only for split groups. For tamely ramified quasi-split groups, we refer to [6, Section 12] for question 2, at least in the equal-characteristic case.

In this paper, we focus on Question 1 above and give a complete answer to this question.

We first show that it suffices to consider quasi-split, semisimple groups of adjoint type (see Sections 2.2, 2.3 for an explanation how to reduce to this case). For such groups, the answer is given in terms of the affine root system and the affine Weyl group of $\mathbf{G}$ and uses the notion of (J, w, δ)-alcove (see Section 3.3), a generalization of the notion of $\mathbf{P}$-alcove introduced in [3] for split groups.

Let $\mathbf{a}$ be the fundamental alcove and $x \in \tilde{W}$. Roughly speaking, the alcove $x\mathbf{a}$ is a (J, w, δ)-alcove if the following two conditions are met: x must satisfy a restriction on its finite part, and the alcove must lie in a certain region of the apartment, which is essentially a union of certain finite Weyl chambers. See Section 3.3 for the precise definition and [3, Section 3] for a visualization.

We denote by $\Gamma_{\mathbb{F}}$ the absolute Galois group of $\mathbb{F}$ and by $\kappa_{\mathbf{G}} \colon \mathbf{G}(\mathbb{L}) \to \pi_1(\mathbf{G})_{\Gamma_{\mathbb{F}}}$ the Kottwitz map; see [8, 13]. Note that $\kappa_{\mathbf{G}}$ also gives rise to maps with source $\tilde{W}$ and source $B(\mathbf{G})$. Likewise, for a Levi subgroup $\mathbf{M}$, we denote by $\kappa_{\mathbf{M}}$ the corresponding Kottwitz map.

Theorem A (Corollary 3.6.1, Theorem 4.4.7). – *Let $b \in \mathbf{G}(\mathbb{L})$ be a basic element, and let $x \in \tilde{W}$. Then $X_x(b) = \varnothing$ if and only if there exists a pair (J, w) such that $x\mathbf{a}$ is a (J, w, δ)-alcove and*

$$\kappa_{\mathbf{M}_J}(w^{-1}x\delta(w)) \notin \kappa_{\mathbf{M}_J}([b] \cap \mathbf{M}_J(\mathbb{L})).$$

We say that an element $x \in \tilde{W}$ lies in the shrunken Weyl chambers if $x\mathbf{a}$ does not lie in the same strip as the base alcove $\mathbf{a}$ with respect to any root direction (cf. Prop. 3.6.5). In this case, we have a more explicit description of the nonemptiness behavior of $X_x(b)$. The answer is given in terms of the map η_δ from $\tilde{W}$ to the finite Weyl group W defined in Section 3.6.

Theorem B (Proposition 3.6.5, Proposition 4.4.9). – *Let $x \in \tilde{W}$ lie in the shrunken Weyl chambers. Let $b \in \mathbf{G}(\mathbb{L})$ be a basic element. Then $X_x(b) \neq \varnothing$ if and only if $\kappa_{\mathbf{G}}(x) = \kappa_{\mathbf{G}}(b)$ and $\eta_\delta(x) \in W - \bigcup_{J \subsetneq S, \delta(J)=J} W_J$.*

Both theorems require that b is a *basic* element. As in [3, Conj. 9.5.1 (b)], we expect that this hypothesis is superfluous for elements x of sufficiently large length (depending on b). However, we are unable to make a precise statement along these lines (this could be seen as formulating a version of Mazur's inequality in the Iwahori case). In applications to the reduction of Shimura varieties, usually the basic case is the most important one: The basic locus is the unique closed Newton stratum (in many cases, this is the supersingular locus), it is the only one where one can hope for a complete geometric description, and it can sometimes be used as a starting point for understanding other Newton strata.

Let us give an overview of the paper. In Section 2 we collect some preliminaries and reduce to the case where $\mathbf{G}$ is quasi-split and semisimple of adjoint type. In Section 3 we prove, imitating the proof given in [3] in the split case, the direction of Theorem A claiming emptiness. In the final Section 4 we prove the non-emptiness statement of the theorem by employing the "reduction method" of Deligne and Lusztig. We show that the notion of (J, w, δ)-alcove is compatible with this reduction. Using some interesting combinatorial properties of affine Weyl groups established by the second-named and third-named authors [7], we are able to reduce the question to the case of $X_x(b)$, where x is of minimal length in its δ-conjugacy class. This case can be handled directly using the explicit description of minimal length elements in [7].