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of affine Deligne-Lusztig varieties*

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## KOTTWITZ-RAPOPORT CONJECTURE ON UNIONS OF AFFINE DELIGNE-LUSZTIG VARIETIES

BY XUHUA HE

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**ABSTRACT.** – In this paper, we prove a conjecture of Kottwitz and Rapoport on a union of (generalized) affine Deligne-Lusztig varieties  $X(\mu, b)_J$  for a  $p$ -adic group  $G$  and its parahoric subgroup  $P_J$ . We show that  $X(\mu, b)_J \neq \emptyset$  if and only if the group-theoretic version of Mazur’s inequality is satisfied. In the process, we obtain a generalization of Grothendieck’s conjecture on the closure relation of  $\sigma$ -conjugacy classes of a twisted loop group.

**RÉSUMÉ.** – Dans cet article nous prouvons une conjecture de Kottwitz et Rapoport sur l’union de variétés de Deligne-Lusztig affines (généralisées)  $X(\mu, b)_J$  pour  $G$  un groupe  $p$ -adique et  $P_J$  son sous-groupe parahorique. Nous montrons que  $X(\mu, b)_J$  est non vide si et seulement si la version de l’inégalité de Mazur pour les groupes est satisfaite. Au cours de la preuve, nous obtenons une généralisation de la conjecture de Grothendieck sur les inclusions des adhérences de classes de  $\sigma$ -conjugaison d’un groupe de lacets tordu.

### Introduction

**0.1.** – The motivation of this paper comes from the reduction of Shimura varieties with a parahoric level structure. On the special fiber, there are two important stratifications:

- Newton stratification, indexed by specific  $\sigma$ -conjugacy classes in the associated  $p$ -adic group  $G$ .
- Kottwitz-Rapoport stratification, indexed by specific double cosets in  $W_J \backslash \tilde{W} / W_J$ , where  $\tilde{W}$  is the Iwahori-Weyl group of  $G$  and  $W_J$  is the Weyl group of the parahoric subgroup  $P_J$ .

A fundamental question is to determine which Kottwitz-Rapoport strata and which Newton strata are nonempty, in other words, to determine the double cosets of  $W_J \backslash \tilde{W} / W_J$  and the subset of  $\sigma$ -conjugacy classes that appear in the reduction of Shimura varieties.

It consists of two parts: local theory and global theory. In this paper, we focus on local theory.

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**0.2.** – In [29] and [23], Pappas and Zhu give a group-theoretic definition of “local models” of Shimura varieties and show that the subset of  $W_J \backslash \tilde{W} / W_J$  for the local model is the admissible set  $\text{Adm}_J(\mu)$  (defined in § 1.5).

The next question is to describe the  $\sigma$ -conjugacy classes arising in the reduction of Shimura varieties. Based on some foundational relations between Newton strata, Kottwitz-Rapoport strata and affine Deligne-Lusztig varieties, we study the set  $X(\mu, b)_J$ , a union of generalized affine Deligne-Lusztig varieties indexed by  $\text{Adm}_J(\mu)$ . It is defined as follows. Let  $L$  be the completion of the maximal unramified extension of a  $p$ -adic field and  $b \in G(L)$ , set

$$X(\mu, b)_J = \{gP_J \in G(L)/P_J; g^{-1}b\sigma(g) \in \bigcup_{w \in \text{Adm}_J(\mu)} P_J w P_J\}.$$

Kottwitz and Rapoport introduced a set  $B(G, \mu)$  of acceptable  $\sigma$ -conjugacy classes, defined by the group-theoretic version of Mazur’s theorem. The main purpose of this paper is to prove the following result, conjectured by Kottwitz and Rapoport in [18] and [24].

**THEOREM A.** – *Suppose that  $G$  splits over a tamely ramified extension of  $F$ . Then  $X(\mu, b)_J \neq \emptyset$  if and only if  $[b] \in B(G, \mu)$ .*

**0.3.** – The direction

$$X(\mu, b)_J \neq \emptyset \Rightarrow [b] \in B(G, \mu)$$

is the group-theoretic version of Mazur’s inequality between the Hodge polygon of an  $F$ -crystal and the Newton polygon of its underlying  $F$ -isocrystal. The case where  $G$  is an unramified group and  $P_J$  is a hyperspecial maximal subgroup, is proved by Rapoport and Richartz in [25, Theorem 4.2]. Another proof is given by Kottwitz in [17]. The case where  $G$  is an unramified group and  $P_J$  is an Iwahori subgroup, is proved in [24, Notes added June 2003, (7)].

The other direction

$$X(\mu, b)_J \neq \emptyset \Leftarrow [b] \in B(G, \mu)$$

is the “converse to Mazur’s inequality” and was proved by Wintenberger in [28] in case  $G$  is quasi-split.

**0.4.** – Another related question is to determine the non-emptiness pattern for a single affine Deligne-Lusztig variety.

If  $G$  is quasi-split and  $P_J$  is a special maximal parahoric subgroup, then the non-emptiness pattern of a single affine Deligne-Lusztig variety is still governed by Mazur’s inequality. It is conjectured and proved for  $G = GL_n$  or  $GS_{p_{2n}}$  by Kottwitz and Rapoport in [18]. It is then proved by Lucarelli [19] for classical split groups and then by Gashi [1] for unramified cases. The general case is proved in [12, Theorem 7.1]. Notice that if  $P_J$  is a special maximal parahoric subgroup and  $\mu$  is minuscule with respect to  $\tilde{W}$ ,  $X(\mu, b)_J$  is in fact a single affine Deligne-Lusztig variety.

If  $P_J$  is an Iwahori subgroup and  $b$  is basic, a conjecture on the non-emptiness pattern (for split groups) is given by Görtz, Haines, Kottwitz, and Reuman in [2] in terms of  $P$ -alcoves in [2] and the generalization of this conjecture to any tamely ramified groups is proved in [4]. The non-emptiness pattern for basic  $b$  and other parahoric subgroups can then be deduced from Iwahori case easily.

However, such information is not useful for the study of  $X(\mu, b)_J$ . The reason is that for  $b$  basic, it is very easy to determine whether  $X(\mu, b)_J$  is empty (by checking the image under Kottwitz map) and for other  $b$ , and non-special parahoric subgroup  $J$ , very little is known about the non-emptiness pattern for a single affine Deligne-Lusztig variety.

**0.5.** – Now we discuss the strategy of the proof of Theorem A. The key ingredients are

- the partial order on  $B(G)$ ;
- some nice properties on the admissible set  $\text{Adm}_J(\mu)$ ;
- the fact that the maximal element in  $B(G, \mu)$  is represented by an element in the admissible set.

We discuss the first ingredient in this subsection and the second and third ingredients in the next subsection.

The starting point is the natural map

$$\Psi : B(\tilde{W}, \sigma) \rightarrow B(G)$$

from the set of  $\sigma$ -conjugacy classes of  $\tilde{W}$  to the set of  $\sigma$ -conjugacy classes of  $G(L)$ . This map is surjective, but not injective in general. However, there exists a natural section of  $\Psi$  given by the straight  $\sigma$ -conjugacy classes of  $\tilde{W}$  (see § 2.2).

On the set of straight  $\sigma$ -conjugacy classes of  $\tilde{W}$ , there is a natural partial order  $\preceq_\sigma$  (defined in § 3.2). On  $B(G)$ , there are two partial orders, given by the closure relation between the  $\sigma$ -conjugacy classes and given by the dominance order of the corresponding Newton polygons. A generalization of Grothendieck conjecture says that the two partial orders on  $B(G)$  coincide. We prove in Theorem 3.1 that

**THEOREM B.** – *For any twisted loop group, the partial order  $\preceq_\sigma$  on the set of straight  $\sigma$ -conjugacy classes coincides with both partial orders on  $B(G)$  via the map  $\Psi : B(\tilde{W}, \sigma) \rightarrow B(G)$ . In particular, the two partial orders on  $B(G)$  coincide.*

The proof is based on the reductive method in [12] à la Deligne and Lusztig, some remarkable combinatorial properties on  $\tilde{W}$  established in [13] and the Grothendieck conjecture for split groups proved by Viehmann in [27].

**0.6.** – By definition,

$$X(\mu, b)_J \neq \emptyset \Leftrightarrow [b] \cap \bigcup_{w \in \text{Adm}_J(\mu)} P_J w P_J \neq \emptyset.$$

Using a similar argument as in the proof of Theorem B, the latter condition is equivalent to  $[b] \in \Psi(\text{Adm}_J(\mu))$ .

Notice that Mazur’s inequality is defined using the dominance order on the Newton polygons. For quasi-split groups, it is easy to see that  $\mu$  is the unique maximal element in  $B(G, \mu)$  with respect to the dominance order. Thus the converse to Mazur’s inequality follows from the coincides between the partial order  $\preceq_\sigma$  on the set of straight  $\sigma$ -conjugacy classes and the dominance order on the Newton polygons. For non quasi-split groups, the maximal element in  $B(G, \mu)$  is harder to understand and we use [14] on the properties of this element.

The proof of Mazur’s inequality is based on two properties of the admissible sets: