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# FINITE GENERATION AND CONTINUITY OF TOPOLOGICAL HOCHSCHILD AND CYCLIC HOMOLOGY

BY Bjørn Ian DUNDAS AND Matthew MORROW

ABSTRACT. – The goal of this paper is to establish fundamental properties of the Hochschild, topological Hochschild, and topological cyclic homologies of commutative, Noetherian rings, which are assumed only to be F-finite in the majority of our results. This mild hypothesis is satisfied in all cases of interest in finite and mixed characteristic algebraic geometry. We prove firstly that the topological Hochschild homology groups, and the homotopy groups of the fixed point spectra $TR^r$, are finitely generated modules (after $p$-completion in the mixed characteristic setting). We use this to establish the continuity of these homology theories for any given ideal. A consequence of such continuity results is the pro Hochschild-Kostant-Rosenberg theorem for topological Hochschild and cyclic homology. Finally, we show more generally that the aforementioned finite generation and continuity properties remain true for any proper scheme over such a ring.

RÉSUMÉ. – Le but de cet article est d'établir des propriétés fondamentales des homologies de Hochschild, de Hochschild topologique et cyclique topologique d'anneaux commutatifs et noethériens, qu'on ne suppose être que F-finis pour la majorité de nos résultats. Cette hypothèse faible est satisfaite en tous cas d'intérêts en géométrie algébrique en caractéristique finie et mixte. Nous démontrons d'abord que les groupes d'homologie de Hochschild topologique, ainsi que les groupes d'homotopie du spectre des points fixés $TR^r$, sont des modules de type fini (après la $p$-complétion dans le cadre de caractéristique mixte). En l'utilisant, nous établissons la continuité de ces homologies pour n'importe quel idéal. Une conséquence de ces résultats de continuité est le théorème de Hochschild-Kostant-Rosenberg pro pour les homologies de Hochschild topologique et cyclique topologique. Finalement, nous démontrons que ces résultats de génération finie et ces propriétés de continuité sont toujours valables pour les schémas propres et lisses sur un tel anneau.

## 1. Introduction

The aim of this paper is to prove fundamental finite generation, continuity, and pro Hochschild-Kostant-Rosenberg theorems for the Hochschild, topological Hochschild, and topological cyclic homologies of commutative, Noetherian rings. As far as we are aware, these are the first general results on the finite generation and continuity of topological


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Hochschild and cyclic homology, despite the obvious foundational importance of such problems in the subject.

The fundamental hypothesis for the majority of our theorems is the classical notion of F-finiteness:

Definition 1.1. – A $\mathbb{Z}_{(p)}$-algebra (always commutative) is said to be *F-finite* if and only if the $\mathbb{F}_p$-algebra $A/pA$ is finitely generated over its subring of $p$-th powers.

This is a mild condition: it is satisfied as soon as $A/pA$ is obtained from a perfect field by iterating the following constructions any finite number of times: passing to finitely generated algebras, localising, completing, or Henselising; see Lemma 3.8.

To state our main finite generation result, we first remark that the Hochschild homology $HH_n(A)$ of a ring $A$ is always understood in the derived sense (see Section 2.2). Secondly, $THH_n(A)$ denotes the topological Hochschild homology groups of a ring $A$, while $TR^r_n(A;p)$ denotes the homotopy groups of the fixed point spectrum $TR^r(A;p)$ for the action of the cyclic group $C_{p^{r-1}}$ on the topological Hochschild homology spectrum $THH(A)$. It is known that $TR^r_n(A;p)$ is naturally a module over the $p$-typical Witt ring $W_r(A)$. Note that $W_1(A) = A$ and $TR^1(A,p) = THH(A)$. The obvious notation will be used for the $p$-completed, or finite coefficient, versions of these theories, and for their extensions to quasi-separated, quasi-compact schemes following [9].

Our main finite generation result is the following, where $A\hat{_p} = \varprojlim_s A/p^sA$ denotes the $p$-completion of a ring $A$:

Theorem 1.2 (see Corol. 4.8). – *Let $A$ be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $n \geq 0$, $r \geq 1$. Then $HH_n(A;\mathbb{Z}_p)$ and $THH_n(A;\mathbb{Z}_p)$ are finitely generated $A\hat{_p}$-modules, and $TR^r_n(A;p,\mathbb{Z}_p)$ is a finitely generated $W_r(A\hat{_p})$-module.*

The key step towards proving Theorem 1.2 is the following finite generation result for the André-Quillen homology of $\mathbb{F}_p$-algebras:

Theorem 1.3 (see Thm. 4.6). – *Let $A$ be a Noetherian, F-finite $\mathbb{F}_p$-algebra. Then the André-Quillen homology groups $D^i_n(A/\mathbb{F}_p)$ are finitely generated for all $n, i \geq 0$.*

Next we turn to "degree-wise continuity" for the homology theories $HH$, $THH$, and $TR^r$, by which we mean the following: given an ideal $I \subseteq A$, we examine when the natural map of pro $A$-modules

$$\{HH_n(A) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(A/I^s)\}_s$$

is an isomorphism, and similarly for $THH$ and $TR^r$. This question was first raised by L. Hesselholt in 2001 [4], who later proved with T. Geisser the $THH$ isomorphism in the special case that $A = R[X_1,\dots,X_d]$ and $I = \langle X_1,\dots,X_s\rangle$ for any ring $R$ [12, §1].

In Section 5.1 we prove the following:

THEOREM 1.4 (see Thm. 5.3). – *Let $A$ be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $I \subseteq A$ an ideal. Then, for all $n \geq 0$ and $v, r \geq 1$, the canonical maps*

$$\{HH_n(A;\mathbb{Z}/p^v) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(A/I^s;\mathbb{Z}/p^v)\}_s$$
$$\{TR^r_n(A;p,\mathbb{Z}/p^v) \otimes_{W_r(A)} W_r(A/I^s)\}_s \longrightarrow \{TR^r_n(A/I^s;p,\mathbb{Z}/p^v)\}_s$$

*are isomorphisms of pro $A$-modules and pro $W_r(A)$-modules respectively.*

Applying Theorem 1.4 simultaneously to $A$ and its completion $\widehat{A} = \varprojlim_s A/I^s$ with respect to the ideal $I$, we obtain Corollary 5.4, stating that both the maps

$$HH_n(A;\mathbb{Z}/p^v) \otimes_A \widehat{A} \longrightarrow HH_n(\widehat{A};\mathbb{Z}/p^v) \longrightarrow \varprojlim_s HH_n(A/I^s;\mathbb{Z}/p^v)$$

are isomorphisms, and similarly for $THH$ and $TR^r$.

Of a more topological nature than such degree-wise continuity statements are spectral continuity, namely the question of whether the canonical map of spectra

$$THH(A) \longrightarrow \operatorname*{holim}_s THH(A/I^s)$$

is a weak equivalence, at least after $p$-completion. The analogous continuity question for $K$-theory was studied for discrete valuation rings by A. Suslin [36], I. Panin [28], and the first author [7], and for power series rings $A = R[[X_1, \dots, X_d]]$ over a regular, F-finite $\mathbb{F}_p$-algebra $R$ by Geisser and Hesselholt [12], with $I = \langle X_1, \dots, X_d \rangle$. Geisser and Hesselholt proved the continuity of $K$-theory in such cases by establishing it first for $THH$ and $TR^r$.

We use the previous degree-wise continuity results to prove the following:

THEOREM 1.5 (see Thm. 5.5). – *Let $A$ be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $I \subseteq A$ an ideal; assume that $A$ is $I$-adically complete. Then, for all $r \geq 1$, the canonical map of spectra*

$$TR^r(A;p) \longrightarrow \operatorname*{holim}_s TR^r(A/I^s;p)$$

*is a weak equivalence after $p$-completion. Similarly for $THH$, $TR$, $TC^r$, and $TC$.*

There are two important special cases in which the results so far can be analysed further: when $p$ is nilpotent, and when $p$ generates $I$. Firstly, if $p$ is nilpotent in $A$, for example if $A$ is a Noetherian, F-finite, $\mathbb{F}_p$-algebra, then Theorem 1.2 – Theorem 1.5 are true integrally, without $p$-completing or working with finite coefficients; see Corollaries 4.9 and 5.6 for precise statements. Secondly, if $I = pA$, then Theorems 1.4 and 1.5 simplify significantly; see Corollary 5.8 for the precise statement and Remark 5.9 for related work.

We present our pro Hochschild-Kostant-Rosenberg (HKR) theorems in Section 5.2. Given a geometrically regular (e.g., smooth) morphism $k \to A$ of Noetherian rings, the classical HKR theorem, combined with Néron-Popescu desingularisation, states that the antisymmetrisation map $\Omega^n_{A/k} \to HH^k_n(A)$ is an isomorphism of $A$-modules for all $n \geq 0$. In Theorem 5.11 we establish its pro analogue: if $I \subseteq A$ is an ideal, then the canonical map of pro $A$-modules

$$\{\Omega^n_{(A/I^s)/k}\}_s \longrightarrow \{HH^k_n(A/I^s)\}_s$$

is an isomorphism for all $n \geq 0$.