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STABLE AND UNSTABLE OPERATIONS IN ALGEBRAIC COBORDISM

BY ALEXANDER VISHIK

ABSTRACT. – We describe additive (unstable) operations from a theory A^* obtained from the Levine-Morel algebraic cobordism by change of coefficients to any oriented cohomology theory B^* (over a field of characteristic zero). We prove that there is 1-to-1 correspondence between operations $A^n \to B^m$ and families of homomorphisms $A^n((\mathbb{P}^\infty)^{\times r}) \to B^m((\mathbb{P}^\infty)^{\times r})$ satisfying certain simple properties. This provides an effective tool of constructing such operations. As an application, we prove that (unstable) additive operations in algebraic cobordism are in 1-to-1 correspondence with the $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$-linear combinations of Landweber-Novikov operations which take integral values on the products of projective spaces. Furthermore, the stable operations are precisely the $\mathbb{L}$-linear combinations of the Landweber-Novikov operations. We also show that multiplicative operations $A^* \to B^*$ are in 1-to-1 correspondence with the morphisms of the respective formal group laws. We construct integral Adams operations in algebraic cobordism, and all theories obtained from it by change of coefficients, extending the classical Adams operations in algebraic K-theory. We also construct symmetric operations and Steenrod operations (à la T. tom Dieck) in algebraic cobordism for all primes. (Only symmetric operations for the prime 2 were previously known to exist.) Finally, we prove the Riemann-Roch Theorem for additive operations which extends the multiplicative case done in [18].

RÉSUMÉ. – Nous décrivons les opérations additives (instables) d'une théorie A^* obtenue par changement de coefficients à partir du cobordisme algébrique de Levine-Morel vers une théorie cohomologique orientée quelconque B^* (sur un corps de caractéristique nulle). Nous établissons une correspondance bijective entre les opérations $A^n \to B^m$ et les familles de morphismes $A^n((\mathbb{P}^\infty)^{\times r}) \to B^m((\mathbb{P}^\infty)^{\times r})$ satisfaisant certaines propriétés simples. Cela fournit une manière effective de construire de telles opérations. Comme application, nous prouvons que les opérations additives (instables) internes au cobordisme algébrique sont en correspondance bijective avec les combinaisons $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$-linéaires des opérations de Landweber-Novikov. Nous montrons également que les opérations multiplicatives $A^* \to B^*$ sont en correspondance bijective avec les morphismes entre les lois de groupes formels respectives. Nous construisons des opérations d'Adams sans dénominateurs en cobordisme algébrique et en toute théorie obtenue à partir du cobordisme algébrique par changement de coefficients, qui étendent les opérations classiques d'Adams en K-théorie. Nous construisons également des opérations symétriques et de Steenrod (à la T. tom Dieck) en cobordisme algébrique, pour tout nombre premier. (Seules les opérations symétriques pour le nombre premier 2 étaient définies auparavant). Enfin, nous prouvons le théorème de Riemann-Roch pour les opérations additives, ce qui généralise le cas multiplicatif traité en [18].

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1. Introduction

In the current article we study operations between oriented cohomology theories (over a field of characteristic zero). In the algebro-geometric context operations were studied by Voevodsky [30], Brosnan [5], Panin-Smirnov [19],[17],[18],[22],[23], and Levine-Morel [15]. By the work of Levine-Morel [15], one has a universal oriented cohomology theory, called algebraic cobordism, and denoted by Ω^*. The universality of Ω^* combined with the reorientation procedure of Panin-Smirnov (following Quillen [21], see also [15, pages 99-105]) permitted to produce the multiplicative operations $\Omega^* \to B^*$ easily and to classify them (in the "invertible" case). In particular, one gets that all such operations are specializations of the Total Landweber-Novikov operation $\Omega^* \to \Omega^*[b_1, b_2, \ldots]$. Previously, the only example of unstable operations (in the algebro-geometric context), the so-called, Symmetric operations (mod 2) were introduced in [24] and [26]. Originally constructed with the aim of producing maps between Chow groups of different quadratic Grassmannians (of the same quadratic form), these operations in algebraic cobordism were successfully applied to the question of rationality of algebraic cycles ([25],[27]), where they provide the only known method to deal with 2-torsion. These operations can be combined into a total one which is a "formal half" of the "negative part" of the Total Steenrod operation (mod 2) in Algebraic Cobordism—see 6.4. The topological counterpart of it was used by Quillen in [21]. Symmetric operations (mod 2) are more subtle than the Landweber-Novikov ones. They lack the 2-primary divisibilities of the latter, and so, in some sense, "plug the gap" between $\mathbb{L}$ and $H_*(MU)$ left by the Hurewicz map, plug 2-adically. To have an integral variant of such statements one would need Symmetric operations for all primes. Unfortunately, the case $p = 2$ was produced by an explicit geometric construction (using $Hilb_2$), and it is unclear how to extend it for other primes. The desire to construct these operations was the main motivation behind the current article. In the end, it appeared that to produce Symmetric operation for $p > 2$ is about as "simple" as to produce all (unstable) additive operations in algebraic cobordism. But to do it, one has to develop some new tools. One needs to understand the internal structure of algebraic cobordism and, more precisely, the way $\Omega^*(X)$ can be described in terms of the restriction of Ω^* to varieties of dimension lower than the dimension of X. This leads to the notion of a *theory of rational type*. Such theories appear to be the same as the *free theories* of Levine-Morel. In particular, all the "standard" theories, like CH, K_0, BP, higher Morava's K-theories $K(n)$ are of this sort. At this stage I should recall that there are two types of cohomology theories in Algebraic Geometry: "large" ones $A^{j,i}$ represented by some spectrum in $\mathbb{A}^1$-homotopy theory, numbered by two indices, and "small" ones A^i, typically, represented by the $(2*,*)$-part of large theories. The Levine-Morel algebraic cobordism Ω^* belongs to the second type and, by the result of Levine ([14], see also [10]), is the $(2*,*)$-part of Voevodsky's MGL. In this article, we work with "small" theories. The fact that Ω^* is a *theory of rational type* is non-trivial. Our proof uses the mentioned comparison result of Levine ([14]). Any theory A^* of rational type on a variety X is described by the values of A^* on varieties of lower dimension. We provide three alternative descriptions here: two in terms of push-forwards, and one in terms of pull-backs—see Subsections 4.1,4.2,4.3. After that it becomes possible to construct operations inductively on dimension.

This enables us to show that an operation can be reconstructed from its action on $(\mathbb{P}^\infty)^{\times r}$, for all r. This is our main result (see Theorem 5.1):

THEOREM 1.1. – *Let A^* be a theory, obtained from Ω^* by change of coefficients, and B^* be any theory in the sense of Definition 2.1. Fix $n, m \in \mathbb{Z}$. Then there is a one-to-one correspondence between additive operations $A^n \xrightarrow{G} B^m$ and families of homomorphisms*

$$A^n((\mathbb{P}^\infty)^{\times l}) \xrightarrow{G} B^m((\mathbb{P}^\infty)^{\times l}), \text{ for } l \in \mathbb{Z}_{\geqslant 0}$$

commuting with pull-backs for:

(i) *the action of $\mathfrak{S}_l$;*
(ii) *the partial diagonals;*
(iii) *the partial Segre embeddings;*
(iv) $(\mathrm{Spec}(k) \hookrightarrow \mathbb{P}^\infty) \times (\mathbb{P}^\infty)^{\times r}, \ \forall r$;
(v) *the partial projections.*

In topology an analogous result was obtained by T. Kashiwabara in [11, Theorem 4.2]. The "multiplicative" variant of our result (Proposition 5.20) says that multiplicative operations correspond to families of homomorphisms as above commuting also with the external products of projective spaces. These results permit to describe and construct operations effectively, as one only needs to define them on $(\mathbb{P}^\infty)^{\times r}$, which is a cellular space. As a first application, we describe all additive (unstable) operations in the Levine-Morel algebraic cobordism. These appear to be exactly those $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$-linear combinations (infinite, in general) of the Landweber-Novikov operations which take "integral" values on $\Omega^*((\mathbb{P}^\infty)^{\times r})$, for all r. This is done in Theorem 6.1:

THEOREM 1.2. – *Let $\psi \in \mathrm{Hom}_{\mathbb{L}}(\mathbb{L}[\overline{\mathrm{b}}], \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q})_{(m-n)}$ be a homomorphism of $\mathbb{L}$-modules. Denote by $S_\psi : \Omega^n \to \Omega^m \otimes_{\mathbb{Z}} \mathbb{Q}$ the respective $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$-linear combination of the Landweber-Novikov operations, i.e., the composition of*

$$\Omega^* \xrightarrow{S_{LN}^{Tot}} \Omega^*[\overline{\mathrm{b}}] \cong \Omega^* \otimes_{\mathbb{L}} \mathbb{L}[\overline{\mathrm{b}}] \xrightarrow{\otimes\psi} \Omega^{*-n+m} \otimes_{\mathbb{Z}} \mathbb{Q}$$

in degree n. Assume that S_ψ satisfies the following integrality condition: $S_\psi(\Omega^n((\mathbb{P}^\infty)^{\times r})) \subset \Omega^m((\mathbb{P}^\infty)^{\times r})$, for all $r \geqslant 0$. Then there exists a unique additive operation $G_\psi : \Omega^n \to \Omega^m$ such that $S_\psi = G_\psi \otimes \mathbb{Q}$. Moreover, every additive operation arises in this way, for a unique ψ. Thus, $\psi \leftrightarrow G_\psi$ is a 1-to-1 correspondence between linear combinations of Landweber-Novikov operations satisfying integrality conditions and integral additive operations.

With the above notation, the stable operations are precisely the G_ψ for $\psi \in \mathrm{Hom}_{\mathbb{L}}(\mathbb{L}[\overline{\mathrm{b}}], \mathbb{L})$, i.e., they are the $\mathbb{L}$-linear combinations of the Landweber-Novikov operations. (See Theorem 3.10 whose proof is much simpler than the above theorem.)

Next, we get a complete description of multiplicative operations from a free theory (in the sense of Levine-Morel) to any other theory in terms of formal group laws. It is given by Theorem 6.9:

THEOREM 1.3. – *Let A^* be a free theory, and B^* be any oriented cohomology theory. The map sending the multiplicative operation $A^* \to B^*$ to the induced homomorphism of formal group laws $(A^*(k), F_A) \to (B^*(k), F_B)$ is a bijection.*