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by

James Arthur

Abstract. — We give a simplified description of a recent classification of automorphic representations of quasisplit orthogonal and symplectic groups. There are three sections, indexed by the three words in the title, which begin with the fundamental notion of an automorphic family of Hecke eigenvalues, and conclude in a description of the main multiplicity formula for the automorphic representations in the discrete spectrum.

***Résumé* (Familles propres, caractères et multiplicités).** — On donne une description simplifiée d'une récente classification des représentations automorphes des groupes orthogonaux et symplectiques quasi-déployés. L'article comprend trois sections qui correspondent aux trois parties du titre. Elles commencent avec la notion fondamentale de famille automorphe de valeurs propres de Hecke et se concluent par une description de la principale formule de multiplicité pour les représentations automorphes du spectre discret.

Foreword

This article is expository. It consists of a short description of the main results of [**A2**], namely a characterization of the automorphic discrete spectrum of a quasisplit orthogonal or symplectic group G. The article [**A3**] also contains a summary of the results of [**A2**]. However, we simplified the discussion there by defining global parameters in terms of the hypothetical global Langlands group L_F. Our focus here will be somewhat different. In particular, we shall formulate the global parameters we need as in the original monograph, simplified somewhat, but still without recourse to the undefined group L_F.

We are assuming for the moment that the field F is global (of characteristic 0). We recall that the global Langlands group L_F is a hypothetical, locally compact

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extension of the global Weil group W_F by a subgroup K_F that is compact, connected and (if we are prepared to be optimistic) even simply connected. It would be characterized by the property that its irreducible, unitary, N-dimensional representations parametrize unitary cuspidal automorphic representations of the general linear group $GL(N)$ over F. However, its existence is far deeper than any theorems now available. The present role of L_F is therefore confined to one of motivation and guidance.

The global parameters ψ in [**A2**] were in fact defined crudely in terms of cuspidal automorphic representations of general linear groups (rather than irreducible finite dimensional representations of the hypothetical group L_F). This leads to a workable substitute $\mathcal{L}_\psi$ for L_F. But as the notation suggests, it has the unfortunate property of being dependent on ψ. We would be better off having a group that at the very least is independent of ψ. I had originally planned to include the construction of such a group in this paper. It is a locally compact group $\widetilde{L}_F^*$ over W_F that is indeed independent of ψ, and which for the purposes of [**A2**] should serve as a substitute for the universal group L_F. It amounts to an extension of the group $\widetilde{L}_{F,\mathrm{reg}}^*$ introduced in [**A2**, §8.5]. However, the construction of $\widetilde{L}_F^*$ is related to questions in base change and automorphic induction that, for me at least, require some further thought. Rather than take the time here, I shall leave it for another paper.

This article will therefore be restricted to our brief survey of results from [**A2**]. It consists of three sections, each devoted to its own general theme. We have chosen the title to reflect these themes, and to draw attention to another difference from the survey [**A3**]. We have tried here to motivate the results from a more elementary and explicit point of view. Each theme leads naturally to the next, until we end in §3 with the global multiplicity formula for G. I hope that the two surveys will be complementary, despite inevitably having much in common. In this article we have emphasized the underlying context of the results (including the role of L_F and its possible substitutes), while [**A3**] was designed more as a guide to their proofs. In particular, there will be no discussion here of the trace formula for G and its stabilization, or the twisted trace formula for $GL(N)$, and its conditional stabilization on which the results still depend.

In §1, we describe automorphic families

$$c = \{c_v : \ v \notin S\}$$

of Hecke eigenvalues for G. The general transfer of these objects is perhaps the most concrete and fundamental manifestation of Langlands's principle of functoriality. However, the endoscopic transfer of Hecke eigenfamilies leads immediately to the more complex question of how automorphic spectra behave under transfer. This question cannot be framed in the absence of further local information. It forces us to provide a corresponding local theory of endoscopic transfer.

In §2, we describe the classification of irreducible representations of a localization $G(F_v)$ of G. These results will be formulated explicitly in terms of irreducible characters, and the transfer factors of Kottwitz, Langlands and Shelstad. We will then be able to state the main global theorem in §3. It gives a decomposition of the automorphic discrete spectrum

$$L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$$

of G in terms of global, "square integrable" parameters $\psi \in \widetilde{\Psi}_2(G)$. The data ψ are the global objects that would be defined naturally in terms of the hypothetical group L_F, but which must in practice be constructed in a more prosaic manner.

The results described in §1–§3 are special cases of Langlands's conjectural theory of endoscopy. They also give special cases of the broader principle of functoriality. However, they occupy a special niche within the general theory. This is because a global parameter $\psi \in \widetilde{\Psi}_2(G)$ is uniquely determined by its associated Hecke eigenfamily

$$c(\psi) = \big\{c_v(\psi) = c(\psi_v) : \ v \notin S\big\},$$

regarded in fact as a family of conjugacy classes in a complex general linear group $GL(N, \mathbb{C})$. In other words, the automorphic representation theory of G is governed by the concrete objects introduced early in §1. This circumstance is also behind the construction of the group $\widetilde{L}^*_F$, which we have postponed for now.

We conclude the introduction with a review of the relevant groups. We take F to be a local or global field of characteristic 0, and G to be a quasisplit, special orthogonal or symplectic group over F. (We assume always that G is "classical", in the sense that it is not an outer twist of the split group $SO(8)$ by a triality automorphism.) For the first three sections of this paper, we follow the conventions from the beginning of [**A3**]. Then G has a complex dual group $\widehat{G}$, and a corresponding L-group

$${}^LG = \widehat{G} \rtimes \Gamma_{E/F}.$$

We are taking $\Gamma_{E/F} = \mathrm{Gal}(E/F)$ to be the Galois group of a suitable finite extension E/F. If G is split, for example, the absolute Galois group $\Gamma = \Gamma_F = \Gamma_{\overline{F}/F}$ acts trivially on $\widehat{G}$, and we often take $E = F$.

There are three general possibilities for G, whose description we take from page 2 of [**A3**]. They correspond to the three infinite families of simple groups $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$, and are as follows.

Type $\mathbf{B}_n$*:* $G = SO(2n+1)$ is split, and $\widehat{G} = Sp(2n, \mathbb{C}) = {}^LG$.

Type $\mathbf{C}_n$*:* $G = Sp(2n)$ is split, and $\widehat{G} = SO(2n+1, \mathbb{C}) = {}^LG$.

Type $\mathbf{D}_n$*:* $G = SO(2n)$ is quasisplit, and $\widehat{G} = SO(2n, \mathbb{C})$. In this case, we can take ${}^LG$ to be the semidirect product of $SO(2n, \mathbb{C})$ with $\Gamma_{E/F}$, where E/F is an arbitrary extension of degree 1 or 2 whose Galois group acts by outer automorphisms on $SO(2n, \mathbb{C})$ (which is to say, by automorphisms that preserve a fixed splitting of $SL(2n, \mathbb{C})$). The nontrivial outer automorphism of $SO(2n, \mathbb{C})$ is induced by conjugation by some element in its complement in $O(2n, \mathbb{C})$.