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A CATEGORICAL APPROACH TO THE STABLE CENTER CONJECTURE

by

Roman Bezrukavnikov, David Kazhdan & Yakov Varshavsky

To Gérard Laumon on his 60th birthday

Abstract. — Let G be a connected reductive group over a local non-archimedean field F. The stable center conjecture provides an intrinsic decomposition of the set of equivalence classes of smooth irreducible representations of $G(F)$, which is only slightly coarser than the conjectural decomposition into L-packets. In this work we propose a way to verify this conjecture for depth zero representations. As an illustration of our method, we show that the Bernstein projector to the depth zero spectrum is stable.

***Résumé* (Une approche catégorique de la conjecture du centre stable).** — Soit G un groupe réductif connexe sur un corps local non archimédien F. La conjecture du centre stable fournit une décomposition intrinsèque de l'ensemble des classes d'équivalence de représentations lisses irréductibles de $G(F)$, qui est seulement un peu plus grossière que la décomposition en L-paquets. Nous proposons dans ce travail une voie de vérification de cette conjecture pour les représentations de profondeur zéro. À titre d'illustration de notre méthode, nous montrons que le projecteur de Bernstein vers le spectre de profondeur zéro est stable.

Introduction

The stable center conjecture. — Let G be a connected reductive group over a local non-archimedean field F, let $R(G)$ be the category of smooth complex representations of $G(F)$, and let Z_G be the Bernstein center of $G(F)$, which is by definition the center of the category $R(G)$. Then Z_G is a commutative algebra over $\mathbb{C}$.

Every $z \in Z_G$ defines an invariant distribution ν_z on $G(F)$, and we denote by Z_G^{st} the set of all $z \in Z_G$ such that the distribution ν_z is stable. The stable center conjecture asserts that Z_G^{st} is a unital subalgebra of Z_G.

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This conjecture is closely related to the local Langlands conjecture. Recall that the local Langlands conjecture asserts that the set of equivalence classes of smooth irreducible representations $\mathrm{Irr}(G)$ of $G(F)$ decomposes as a disjoint union of so-called L-packets. By definition, we have a natural homomorphism $z \mapsto f_z$ from Z_G to the algebra of functions $\mathrm{Fun}(\mathrm{Irr}(G), \mathbb{C})$. A more precise version of the stable center conjecture asserts that Z_G^{st} consists of all $z \in Z_G$ such that the function f_z is constant on each L-packet.

In other words, the local Langlands conjecture allows a more precise formulation of the stable center conjecture. Conversely, if $Z_G^{st} \subset Z_G$ is known to be a subalgebra, then we can decompose $\mathrm{Irr}(G)$ by characters of Z_G^{st}, and conjecturally this decomposition is only slightly coarser than the decomposition by L-packets. Thus, the stable center conjecture can be thought both as a supporting evidence and as a step in the proof of the local Langlands conjecture.

As follows from results of Bernstein and Moy-Prasad, the category $R(G)$ decomposes as a direct sum $R(G) = R(G)^0 \oplus R(G)^{>0}$, where $R(G)^0$ (resp. $R(G)^{>0}$) consists of those representations π, all of whose irreducible subquotients have depth zero (resp. positive depth). Therefore the Bernstein center Z_G decomposes as a direct sum of centers $Z_G = Z_G^0 \oplus Z_G^{>0}$. In particular, we have an embedding $Z_G^0 \hookrightarrow Z_G$, which identifies the unit element of Z_G^0 with the projector to the depth zero spectrum $z^0 \in Z_G$. Set $Z_G^{st,0} := Z_G^0 \cap Z_G^{st}$.

The depth zero stable center conjecture asserts that $Z_G^{st,0} \subset Z_G^0$ is a unital subalgebra. In particular, it predicts the stability of the projector $z^0 \in Z_G$.

The main goal of this work is to outline an approach to a proof of the depth zero stable center conjecture. As an illustration of our method, we prove an explicit formula for the Bernstein projector z^0, and deduce its stability. More precisely, we do it when G is a split semisimple simply connected group, and F is a local field of a positive but not very small characteristic.

Our approach. — Our strategy is to construct explicitly many elements z of $Z_G^0 \subset Z_G$, whose span is a subalgebra, and to prove that these elements are stable and generate all of $Z_G^{st,0}$. Here by "explicitly", we mean to describe both the invariant distribution ν_z on $G(F)$ and the function f_z on $\mathrm{Irr}(G)$.

To carry out our strategy, we construct first a categorical analog $\mathcal{Z}(LG)$ of the Bernstein center Z_G. Then we observe that a version of the Grothendieck "sheaf-function correspondence" associates to each Frobenius equivariant object $\mathcal{F} \in \mathcal{Z}(LG)$ an element of the Bernstein center $[\mathcal{F}] \in Z_G$. Thus, to construct elements of Z_G, it suffices to construct Frobenius-equivariant objects of $\mathcal{Z}(LG)$.

In order to construct elements of $\mathcal{Z}(LG)$, we construct first a categorical analog $\mathcal{Z}_{\mathbf{I}^+}(LG)$ of Z_G^0 and a categorical analog $\mathcal{A} : \mathcal{Z}_{\mathbf{I}^+}(LG) \to \mathcal{Z}(LG)$ of the embedding $Z_G^0 \hookrightarrow Z_G$. Then we apply $\mathcal{A}$ to monodromic analogs of Gaitsgory central sheaves.

Roughly speaking, we define $\mathcal{A}$ to be the composition of the averaging functor $\mathrm{Av}_{\mathrm{Fl}}$, where Fl is the affine flag variety of G, and the functor of "derived $\widetilde{W}$-skew-invariants", where $\widetilde{W}$ is the affine Weyl group of G. This construction is motivated by an analogous finite-dimensional result, proven in [**BKV1**]. However, in the affine case one has to overcome many technical difficulties.

Bernstein projector to the depth zero spectrum. — To illustrate our method, we provide a geometric construction of the Bernstein projector $z^0 \in Z_G$. More precisely, we construct a class $\langle A\rangle$ in the Grothendieck group version of $\mathcal{Z}(LG)$ and show that the corresponding element of Z_G is z^0. Then we show that the restriction $\nu_{z^0}|_{G^{rss}(F)}$ is locally constant and prove an explicit formula, which we now describe.

Let I^+ be the pro-unipotent radical of the Iwahori subgroup of $G(F)$, let μ^{I^+} be the Haar measure on $G(F)$ normalized by the condition that $\mu^{I^+}(I^+) = 1$, and let $\phi_{z^0} \in C^\infty(G(F))$ be such that $\nu_{z^0}|_{G^{rss}(F)} = \phi_{z^0}\mu^{I^+}$.

For each $\gamma \in G^{rss}(F)$, we denote by Fl_γ be the corresponding affine Springer fiber. The affine Weyl group $\widetilde{W}$ acts on each homology group $H_i(\mathrm{Fl}_\gamma) = H^{-i}(\mathrm{Fl}_\gamma, \mathbb{D}_{\mathrm{Fl}_\gamma})$, where $\mathbb{D}_{\mathrm{Fl}_\gamma}$ is the dualizing sheaf. Consider the Tor-groups $\mathrm{Tor}_j^{\widetilde{W}}(H_i(\mathrm{Fl}_\gamma), \mathrm{sgn})$, where by sgn we denote the sign-character of $\widetilde{W}$. Each $\mathrm{Tor}_j^{\widetilde{W}}(H_i(\mathrm{Fl}_\gamma), \mathrm{sgn})$ is a finite-dimensional $\overline{\mathbb{Q}_l}$-vector space, equipped with an action of the Frobenius element. One of the main results in this paper is the following identity

$$\phi_{z^0}(\gamma) = \sum_{i,j} (-1)^{i+j}\, \mathrm{Tr}(\mathrm{Fr}, \mathrm{Tor}_j^{\widetilde{W}}(H_i(\mathrm{Fl}_\gamma), \mathrm{sgn})). \tag{0.1}$$

Using formula (0.1) and a group version of a theorem of Yun [**Yun**], we show that $\nu_{z^0}|_{G^{rss}(F)}$ is stable. Note that the proof of Yun is global, while all the other arguments are purely local.

Remark. — Though ∞-categories are not needed for the construction of $\langle A\rangle$, we need them in order to prove the formula (0.1). Moreover, the structure of formula (0.1) indicates why the ∞-categories appears here. The shape of the formula suggests a possibility to write the right hand side of (0.1) as the trace of Frobenius on the "derived skew-coinvariants" $R\Gamma(\mathrm{Fl}, \mathbb{D}_{\mathrm{Fl}_\gamma})_{\widetilde{W},\mathrm{sgn}}$. However, the functor of "derived skew-coinvariants" is defined as a homotopy colimit, and it can not be defined in the framework of derived categories. Therefore one has to pass to stable ∞-categories.

Plan of the paper. — In Section 1 we study derived categories of constructible sheaves on a certain class of ind-schemes and ind-stacks, which we call admissible. This class includes some infinite-dimensional ind-stacks, which are not algebraic. We also construct a certain geometric 2-category, whose ∞-version is used later.

In Section 2, we apply the formalism of Section 1 to the case of loop groups LG and related spaces in order to construct a categorical analog of the Hecke algebra.

Section 3 deals with the stable center conjecture. Namely, we formulate and discuss the stable center conjecture in subsection 3.1, categorify various objects from subsection 3.1 in subsections 3.2-3.3, and describe our (conjectural) approach to the depth zero stable center conjecture in subsections 3.4-3.5.

The results in subsections 3.2-3.3 are given without complete proofs, and details will appear in the forthcoming paper [**BKV2**]. To emphasise this fact, we write *"Theorem"* instead of *Theorem*.