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LINEAR MODELS FOR REDUCTIVE GROUP ACTIONS ON AFFINE QUADRICS

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RÉSUMÉ. — Nous étudions les actions des groupes réductifs sur les quadriques affines complexes dont le quotient est de dimension 1. Une telle action est dite linéarisable si elle est équivalente à la restriction d'une action linéaire orthogonale dans l'espace affine ambiant de la quadrique. Une action linéaire satisfait à certaines conditions topologiques. Nous recherchons si ces conditions sont valables pour des actions générales. Si c'est le cas, il est naturel de se demander si une action donnée possède un modèle linéaire, c'est-à-dire si il existe une action linéaire avec les mêmes types d'orbites et avec des représentations slices équivalentes. Nous montrons qu'un modèle linéaire existe si l'action a un point fixe ou si le groupe d'isotropie principal est connexe. Enfin, nous faisons une classification de toutes les actions linéaires dont le quotient est de dimension 1.

ABSTRACT. — We study reductive group actions on complex affine quadrics with 1-dimensional quotient. Such an action is called linearizable if it is equivalent to the restriction of a linear orthogonal action in the ambient affine space of the quadric. A linear action on the quadric satisfies certain topological conditions. We examine whether these condititons also hold for general actions. In case they do it is natural to ask whether a given action has a linear model, i.e., whether there is a linear action with the same orbit types and equivalent slice representations. We show that a linear model exists if the action has a fixed point or if the principal isotropy group is connected. Finally, we classify all linear actions with 1-dimensinal quotient.

1. Introduction

1.1. — Let $Q_n := \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = 1\} \subset \mathbb{C}^{n+1}$ denote the n-dimensional affine quadric over the field of complex numbers $\mathbb{C}$. Let G be a (linear) algebraic group. Every orthogonal representation $\rho : G \to \mathrm{O}_{n+1}(\mathbb{C})$ determines an action of G on Q_n. These actions

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— we call them *linear actions* — are well understood. Of course, the geometry of the situation does not change if we replace a given action by a conjugate one within the group $\operatorname{Aut} Q_n$ of algebraic automorphisms of Q_n. We call an action of G on Q_n *linearizable* if it is conjugate to a linear action of G on Q_n.

1.2. — The case $n = 1$ is easy. Here

$$Q_1 \cong \dot{\mathbb{C}} = \mathbb{C} \setminus \{0\} \quad \text{and} \quad \operatorname{Aut} Q_1 \cong \mathbb{C}^* \rtimes \mathbb{Z}_2 = \mathrm{O}_2,$$

and so every group action is linear. The situation changes dramatically for $n \geq 2$. In these cases $\operatorname{Aut} Q_n$ can be given the structure of an infinite dimensional algebraic group, see [11]. More is known only for $n = 2$ where this group can be written as an amalgamated product, see [10, p. 94]. The following example illustrates that $\operatorname{Aut} Q_n$ is indeed very big even for small n and shows that unipotent group actions need not be linearizable. Consider Q_3, which we identify with $\mathrm{SL}_2(\mathbb{C})$. Choose a U-invariant regular function f on $\mathrm{SL}_2(\mathbb{C})$, where U denotes the subgroup of matrices with 1's in the diagonal and 0 in the lower left entry. Consider the following action of the additive group $\mathbb{C}^+$ on SL_2 :

$$t \cdot h := \begin{pmatrix} 1 & tf(h) \\ 0 & 1 \end{pmatrix} \cdot h \cdot \begin{pmatrix} 1 & -tf(h) \\ 0 & 1 \end{pmatrix},$$

where $t \in \mathbb{C}^+$ and $h \in \mathrm{SL}_2$. We claim that this action is not linearizable as soon as f is not a constant. In fact, the linear actions of $\mathbb{C}^+$ on SL_2 are easily classified. Under the double cover $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$ the SO_4-action on Q_3 corresponds to the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on SL_2 given by $(g, g') \cdot h = ghg'^{-1}$. Thus a linear action $\mathbb{C}^+ \to \mathrm{SO}_4$ on Q_3 is given by the corresponding morphism $\mathbb{C}^+ \to \mathrm{SL}_2 \times \mathrm{SL}_2$ as an action on SL_2. It follows that such an action must be equivalent to either the trivial action, the one given by conjugation or the one given by left (or right) multiplication. It is now straightforward that of these only the trivial action or the one given by conjugation can be equivalent to the action defined above, and that such an equivalence is only possible if the function f is constant.

1.3. — Because of the previous example we restrict our attention to reductive groups G, i.e., to groups which don't have any non-trivial unipotent normal subgroups. (Equivalently, every rational representation of G is completely reducible.)

Linearization problem : *Is every action of a reductive group on an affine quadric linearizable ?*

So far no example of a non-linearizable reductive group action on Q_n is known. However, we do not believe that every such action is linearizable, except under certain «smallness» assumptions. For example, every reductive group action on Q_2 is linearizable. This follows from the structure theorem for $\operatorname{Aut} Q_2$ mentioned above. We will show among other things that linearization is possible for actions for which the only invariant regular functions on Q_n are the constants, see § 2. Therefore, the classification of these cases is achieved by classifying all orthogonal representations (V, G) for which the ring of invariant functions $\mathcal{O}(V)^G$ is generated by the invariant quadratic form.

1.4. — In case linearization holds the G-action has to satisfy certain topological conditions, e.g. the generic orbit of G on $X = Q_n$ has to be closed. Moreover, every slice representation (N_x, G_x) has to be orthogonal, where $x \in X$ is a point on a closed orbit, G_x is the stabilizer of x and $N_x = \mathrm{T}_x X / \mathrm{T}_x Gx$ is the normal space to the orbit. This follows from the fact that these properties hold for orthogonal representations, see [22, § 5]. This leads to the following

Definition. — An orthogonal representation (V, G) is called a *linear model* for an action of G on the quadric $X = Q_n$ if X has the same orbit types and equivalent slice representations as the quadric

$$Q_V := \{v \in V \mid (v, v) = 1\} \subset V$$

with the linear G-action.

1.5. — The aim of this paper is to study the topology of a connected reductive group action on an affine quadric X under the assumption that the ring of invariants has (Krull-) dimension 1, i.e., that the algebraic quotient $X/\!\!/G$ (see 1.9) is 1-dimensional. It turns out that for our results it is enough to assume that X is an irreducible, smooth affine variety which is homotopy equivalent to a real sphere.

Proposition 1. — *Under the assumptions above we have :*

(1) $X/\!\!/G \cong \mathbb{A}$, *the affine line.*

(2) *There are two points* $y_1, y_2 \in \mathbb{A}$ *such that the principal stratum is* $\mathbb{A} \setminus \{y_1, y_1\}$.

(3) *The generic fiber of the quotient map (i.e., the fiber over the principal stratum) is a G-orbit, which means that the generic orbit is closed.*

This is proved in sections 3.2 and 3.4.

1.6. — The results in PROPOSITION 1 are obvious for linear actions on quadrics, or, more generally, if there is a linear model. We believe that the assumptions in 1.5 insure the existence of a linear model. However, we have been able to prove existence only under additional hypotheses.

PROPOSITION 2. — *Under the assumptions of* 1.5 *a linear model exists in the following cases :*

(1) *The G-action on X has a fixed point.*

(2) *The principal isotropy group of the action is connected, and the dimension of the slice representations is* > 2.

This is proved in sections 4.8 and 4.9.

1.7. — The analogous situation of compact group actions on real spheres has been studied extensively. For example, BOREL, MONTGOMERY and SAMELSON classified all transitive compact group actions on spheres (see [19], [2] and [3]). The case of orbit space dimension 1 has been analyzed by WANG [25] and ASOH [1]. Their results are essential in our approach.

1.8. — We have been guided by the work of KRAFT, LUNA and SCHWARZ on the linearization problem for reductive group actions on affine space $\mathbb{C}^n$. In this classical setting the question is whether a given action is equivalent to a representation, and these authors have tackled the problem under the assumption that the quotient dimension is equal to 1, see [14] and [16]. (Note that actions with quotient dimension 0 are linearizable by Luna's slice theorem.) They first prove with topological methods the existence of a fixed point and then compare the tangent representation at this point with the given action. Although linearization holds in many cases, the first non-linearizable actions on $\mathbb{C}^n$ were discovered by SCHWARZ [23] in this context. Moreover, using the results of SCHWARZ, KNOP [12] proved that every non-commutative, connected reductive group has non-linearizable actions on some $\mathbb{C}^n$. Our approach to the linearization problem on quadrics is the analogon to the one taken in [16]. There the fixed point gives a linear model as the tangent representation at this point. Here we have to carry the topological analysis much further to show that a linear model exists. In section 5.1 we classify all these models, i.e., all linear actions on quadrics with 1-dimensional quotient. This classification will be used in a subsequent paper to show that the existence of a linear model suffices to prove that linearization holds.

1.9. — To conclude this introduction, we state the conventions and notation valid in this paper as well as some general facts. Our varieties