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ON THE MOTION OF A SMALL BODY IMMERSED IN A TWO-DIMENSIONAL INCOMPRESSIBLE PERFECT FLUID

by Olivier Glass, Christophe Lacave & Franck Sueur

Abstract. — In this paper we prove that the motion of a solid body in a two dimensional incompressible perfect fluid converges, when the body shrinks to a point with fixed mass and circulation, to a variant of the vortex-wave system where the vortex, placed in the point occupied by the shrunk body, is accelerated by a lift force similar to the Kutta-Joukowski force of the irrotational theory.

Résumé (*Sur le mouvement d'un petit corps solide immergé dans un fluide parfait incompressible en deux dimensions*)

Dans cet article nous prouvons que le mouvement d'un petit corps solide immergé dans un fluide parfait incompressible en deux dimensions converge, quand le corps se rétrécit en un point, avec sa masse et sa circulation fixées, vers une variante du système « Euler+point vortex » où le vortex, placé au point où le solide a rétréci, est accéléré par un terme de force de portance similaire à la force de Kutta-Joukowski de la théorie irrotationnelle.

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1. Introduction

In this paper we consider the motion of a small solid body in a planar ideal fluid, and the limit behaviour of the system as the solid body is reduced to a point.

Let us first describe the equations when the solid has a fixed size. Let $\mathcal{S}_0$ be a closed, bounded, connected and simply connected subset of the plane with smooth boundary. We assume that the body initially occupies the domain $\mathcal{S}_0$ and rigidly moves so that at time t it occupies an isometric domain denoted by $\mathcal{S}(t)$. We denote $\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t)$ the domain occupied by the fluid at time t starting from the initial domain $\mathcal{F}_0 := \mathbb{R}^2 \setminus \mathcal{S}_0$.

The equations modelling the dynamics of the system then read

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \text{ for } x \in \mathcal{F}(t), \tag{1}$$

$$\operatorname{div} u = 0 \text{ for } x \in \mathcal{F}(t), \tag{2}$$

$$u \cdot n = u_{\mathcal{S}} \cdot n \text{ for } x \in \partial\mathcal{S}(t), \tag{3}$$

$$\lim_{|x|\to\infty} |u| = 0, \tag{4}$$

$$mh''(t) = \int_{\partial\mathcal{S}(t)} pn\, ds, \tag{5}$$

$$\mathcal{J}r'(t) = \int_{\partial\mathcal{S}(t)} (x - h(t))^{\perp} \cdot pn\, ds, \tag{6}$$

$$u|_{t=0} = u_0 \text{ for } x \in \mathcal{F}_0, \tag{7}$$

$$h(0) = h_0,\ h'(0) = \ell_0, r(0) = r_0. \tag{8}$$

Here $u = (u_1, u_2)$ and p denote the velocity and pressure fields, m and $\mathcal{J}$ denote respectively the mass and the moment of inertia of the body while the fluid is supposed to be homogeneous of density 1, to simplify the notations. When $x = (x_1, x_2)$ the notation $x^{\perp}$ stands for $x^{\perp} = (-x_2, x_1)$, n denotes the unit normal vector pointing outside the fluid, $h'(t)$ is the velocity of the center of mass $h(t)$ of the body and $r(t)$ denotes the angular velocity of the rigid body. Finally we denote by $u_{\mathcal{S}}$ the velocity of the body:

$$u_{\mathcal{S}}(t, x) = h'(t) + r(t)(x - h(t))^{\perp}. \tag{9}$$

Since $\mathcal{S}(t)$ is the position occupied by a rigid body there exists a rotation matrix

$$Q(t) := \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}, \tag{10}$$

such that the position $\eta(t,x) \in \mathcal{S}(t)$ at the time t of the point fixed to the body with an initial position x is

$$\eta(t,x) := h(t) + Q(t)(x - h_0). \tag{11}$$

The angle θ satisfies

$$\theta'(t) = r(t),$$

and we choose $\theta(t)$ such that $\theta(0) = 0$.

The Equations (1) and (2) are the incompressible Euler equations, the condition (3) means that the boundary is impermeable and the Equations (5) and (6) are the Newton's balance law for linear and angular momenta.

For the study of ideal flow, an important quantity is the vorticity $w := \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$, satisfying the transport equation:

$$\frac{\partial w}{\partial t} + (u \cdot \nabla) w = 0 \text{ for } x \in \mathcal{F}(t). \tag{12}$$

One has the following result concerning the Cauchy problem for the above system, the initial position of the solid being given. This result describes weak solutions, extending results concerning the fluid alone. It considers the case when vorticity belongs in L^p as in DiPerna and Majda [7] and includes weak solutions with bounded vorticity as in Yudovich [13].

THEOREM 1. — *Let $p \in (2, +\infty]$. For any $u_0 \in C^0(\overline{\mathcal{F}_0}; \mathbb{R}^2)$, $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, such that:*

$$\operatorname{div} u_0 = 0 \ \text{in} \ \mathcal{F}_0 \quad \text{and} \ u_0 \cdot n = (\ell_0 + r_0 (x - h_0)^\perp) \cdot n \ \text{on} \ \partial \mathcal{S}_0, \tag{13}$$

$$w_0 := \operatorname{curl} u_0 \in L^p_c(\overline{\mathcal{F}_0}), \tag{14}$$

$$\lim_{|x| \to +\infty} u_0(x) = 0,$$

there exists a solution (h', r, u) of (1)–(8) in $C^1(\mathbb{R}^+; \mathbb{R}^2 \times \mathbb{R}) \times C^0(\mathbb{R}^+, W^{1,p}(\mathcal{F}(t)))$ with $\partial_t u, \nabla p \in L^\infty_{loc}(\mathbb{R}^+, L^q(\mathcal{F}(t)))$ for any $q \in (1,p]$ when $p < \infty$ and in $C^1(\mathbb{R}^+; \mathbb{R}^2 \times \mathbb{R}) \times L^\infty_{loc}(\mathbb{R}^+, \mathcal{LL}(\mathcal{F}(t)))$ with $\partial_t u, \nabla p \in L^\infty_{loc}(\mathbb{R}^+, L^q(\mathcal{F}(t)))$ for any $q \in (1, +\infty)$ when $p = \infty$.

Moreover such a solution satisfies that for all $t > 0$, $w(t) := \operatorname{curl}(u(t)) \in L^p_c(\overline{\mathcal{F}(t)})$, it is energy-conserving in the sense of Proposition 4 and $\|w(t,\cdot)\|_{L^q(\mathcal{F}(t))}$ (for any $q \in [1,p]$), $\int_{\mathcal{F}(t)} w(t,x)\,dx$ and $\int_{\partial \mathcal{S}(t)} u \cdot \tau \, ds$ are preserved over time.

Finally when $p = \infty$, the solution is unique.

This result is proven in [10]. For the sake of self-containedness, we give a short proof of it in appendix. The notation $\mathcal{LL}(\Omega)$ refers to the space of log-Lipschitz functions on Ω, that is the set of functions $f \in L^\infty(\Omega)$ such that

$$\|f\|_{\mathcal{LL}(\Omega)} := \|f\|_{L^\infty(\Omega)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|(x-y)(1 + \ln^- |x-y|)|} < +\infty. \tag{15}$$