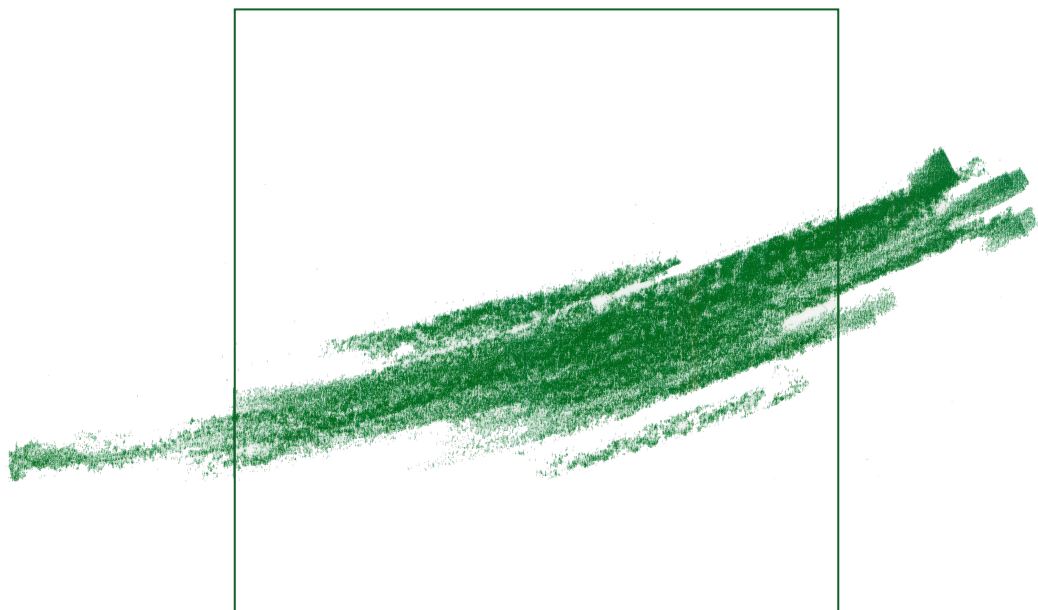


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C O L L E C T I O N S M F

Integral geometry from Buffon to geometers of today

Rémi LANGEVIN



23

**INTEGRAL GEOMETRY FROM BUFFON
TO GEOMETERS OF TODAY**

Rémi Langevin

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