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# *NATURPHILOSOPHIE* AND ITS ROLE IN RIEMANN'S MATHEMATICS

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ABSTRACT. — This paper sets out to examine some of Riemann's papers and notes left by him, in the light of the "philosophical" standpoint expounded in his writings on *Naturphilosophie*. There is some evidence that many of Riemann's works, including his *Habilitationsvortrag* of 1854 on the foundations of geometry, may have sprung from his attempts to find a unified explanation for natural phenomena, on the basis of his model of the ether.

Keywords: ether theory, complex function theory, Riemannian differential geometry.

RÉSUMÉ. — LE RÔLE DE LA *NATURPHILOSOPHIE* DANS LES TRAVAUX MATHÉMATIQUES DE RIEMANN. Dans cet article, nous proposons une lecture de certains mémoires et notes de Riemann à la lumière du point de vue «philosophique» qu'il a développé dans ses écrits sur la *Naturphilosophie*. Il apparaît que l'origine de nombreux travaux de Riemann, y compris l'*Habilitationsvortrag* de 1854 sur les fondements de la géométrie, peut être trouvée dans sa tentative d'explication unitaire des phénomènes naturels sur la base de son modèle de l'éther.

## INTRODUCTION

Riemann's writings on *Naturphilosophie*[1] can be regarded as the result of his attempt to find a unified, mathematical explanation of various physical phenomena such as gravitation, electricity, magnetism and light. They also allow us to include some of his better known papers — such as his *Inauguraldissertation* [1851], his *Habilitationsvortrag* [1854b] and other papers on physical subjects as well — in a wide-ranging research program.

[1] Heinrich Weber gathered these with others manuscripts of Riemann on philosophical subjects, such as psychology, metaphysics and gnosiology, and published them in Riemann's collected works under the title *Fragmente philosophischen Inhalts* [Riemann 1876a].

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As Klein once said, Riemann's work was characterized by his continual attempt to put "*in mathematical form a unified formulation of the laws which lie at the basis of all natural phenomena*" [1894, p. 484]. Klein did not hesitate to claim that "*the origins of Riemann's pure mathematical developments*" lay in this research which, in Riemann's words, was at a certain stage his own "*main work*".

Searching for a mathematical description of the known physical phenomena, Riemann thought of space as pervaded by substance (*Stoff* [2]), and in a section of his *Fragmente* he considered the state of a single particle of substance and analysed locally the space around it [Riemann 1853].

This passage from "local" to "global" constitutes the basic method used by Riemann in some of his most important works in geometry as well as in analysis and physics. In analytical terms this corresponds to the analytical continuation of a complex function. This is "*a well known theorem*" [Riemann 1857b, p. 88] which is at the basis of the "*new method*" he set up in his thesis. This "*method*" [Riemann 1851, p. 37–39] could be applied to Abelian functions, as he did in [1857b], and also "*in its essential lines*" to "*every function which satisfies a linear differential equation with algebraic coefficients*" [1857a, p. 67]. Accordingly, in this paper he studied the transcendental functions defined by the hypergeometric differential equation "*almost without calculations*" [*Werke*, p. 85] and "*and in their totality*" on the complex sphere.

The same point of view inspired his *Habilitationsvortrag* where he defined metrics on manifolds by using the linear element d$s$. In particular, Riemann stated that "*questions about the immeasurably large are idle questions for the explanation of Nature. But the situation is quite different with questions about the immeasurably small*" [Riemann 1854b/1979, p. 151]. As Riemann explained in the introduction to the first course he gave in Göttingen as a *Privatdozent*, the laws for all space could be deduced by integrating partial differential equations expressing some "*elementary*" principles valid for infinitely small portions of space.[3]

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[2] Instead of this, in his later lectures on gravitation, electricity and magnetism Riemann preferred to use the term ether.

[3] "*Wahre Elementargesetze können nur im Unendlichkleinen, nur für Raum — und Zeitpunkte stattfinden. Solche Gesetze aber werden im Allgemeinen partielle Differentialgleichungen sein, und die Ableitung der Gesetze für ausgedehnte Körper und Zeiträume aus ihnen erfordert die Integration derselben. Es sind also Methode*

Such a research method had already been announced by Riemann himself in a rather cryptic way in 1850. When lecturing at the *Pädagogische Seminar* he noticed that it was possible to formulate a mathematical theory by moving from elementary principles toward general laws valid in all of a given continuous space without distinguishing between gravity, electricity, magnetism and equilibrium of heat.[4]

As Klein pointed out, the method of studying functions on the basis of their behaviour in the infinitely small had a physical counterpart in the concept of a line of force. Moreover, Klein suggested a kind of dualism between Riemann's mathematical thought and Faraday's concept of action by contact, writing that: "*If I may dare to proceed with so forceful the analogy, then I shall say that Riemann in the field of mathematics and Faraday in the field of physics are parallel*" [Klein 1894, p. 484].

Supporting Klein's point of view, in *Raum Zeit Materie* Weyl stated that the passage from Euclidean to Riemannian geometry "*is founded in principle on the same idea as that which led from physics based on action at a distance to physics based on infinitely near action*" [1919a/1922, p. 91]. In fact, according to Weyl:

"The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts *is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry, and, indeed, the mainspring of all the eminent work of Riemann, in particular, that dealing with the theory of complex functions*" [1919a/1922, p. 92].

## 1. ON THE SOURCE OF RIEMANN'S ANALYTICAL WORK

Riemann introduced his ideas on complex function theory in his 1851 paper which concluded his studies at Göttingen. Riemann's starting point

*nöthig, durch welche man aus den Gesetzen im Unendlichkleinen diese Gesetze im Endlichen ableitet, und zwar in aller Strenge ableitet, ohne sich Vernachlässigungen zu erlauben. Denn nur dann kann man sie an der Erfahrung prüfen*" [Riemann 1869, p. 4].

[4] "*So z.B. lässt sich eine vollkommen in sich abgeschlossene mathematische Theorie zusammenstellen, welche von den für die einzelnen Punkte geltenden Elementargesetzen bis zu den Vorgängen in dem uns wirklich gegebenen continuirlich erfüllten Raume fortschreitet, ohne zu scheiden, ob es sich um die Schwerkraft, oder die Electricität, oder den Magnetismus, oder das Gleichgewicht der Wärme handelt*" (in [Dedekind 1876, p. 545]).

was given by the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1.1}$$

which have to be satisfied by the function $w = u + iv$ of a variable $z = x + iy$. From (1.1) he deduced the equations $\Delta u = 0$, $\Delta v = 0$ which are the basis for investigating the properties of the functions $u$ and $v$ [Riemann 1851, p. 7].

As Prym was to write to Klein after Riemann's death,[5] since his student days Riemann had attributed great importance to equations (1.1) for the continuation of a function from one complex domain to another. According to him, equations (1.1) explain why correct results can be obtained even when working with divergent series, as Euler repeatedly did.

It is a well known fact that Riemann's complex function theory is deeply connected with potential theory in two dimensions — a theory he was well acquainted with. Indeed, as a student Riemann had followed Weber's lectures in 1849 and the following year he participated in the physics seminar jointly founded and led by Gauss and Weber. Gauss himself had developed the theory of the Laplace equation in a paper of 1839. He had determined the potential function in different cases and, in particular, he had studied the problem of the distribution of masses or electric charges on a closed surface $S$, assuming the potential to be constant on $S$.

From a mathematical point of view, this reduced the problem to minimizing the following integral

$$J = \int_V |\operatorname{grad} u|^2 \, \mathrm{d}v.$$

[5] "*Nach einer Mittheilung, die mir Riemann in Frühjahre 1865 während meines Pisaner Aufenthalts machte, ist derselbe zu einer Theorie der Functionen einer verändlichen complexen Grösse durch die Beobachtung gekommen, dass Beziehungen zwischen Functionen, die durch Entwicklung der betreffenden Functionen in Reihen erhalten worden, bestehen bleiben, auch wenn man über die Convergenzgebiete der darstellenden Reihen hinausging, und dass man in vielen Fällen richtige Resultate erhält, wenn man, wie Euler z.B. es wiederholt getan, mit divergenten Reihen operiert. Er frug sich dann, was denn eigentlich die Function aus dem einen Gebiete in das andere fortgesetzt, und gelangte zu der Einsicht, dass dies die partielle Differentialgleichung thue. Dirichlet, mit dem er den Gegenstand besprach, stimmte dieser Ansicht vollständig bei; es fällt also diese Idee wohl noch in die Studienjahre Riemanns, vor die Auffassung seiner Inauguraldissertation*". This letter from February 6, 1882 is kept in Klein's *Nachlass* [11, 383].

Since $J > 0$, "*a [homogeneous] distribution must necessarily exist, so that the integral J has a minimum*", Gauss wrote [1839, p. 233]. This argument was used systematically by Dirichlet in his lectures on the forces which are inversely proportional to the square of distance.

Dirichlet [1876, p. 127] faced the problem of proving that a function with continuous first partial derivatives on a given bounded domain, which satisfies the Laplace equation within the domain and has given values on the boundary, always exist. Dirichlet's existence proof of the solution of the "Dirichlet problem" was based on the fact that the minimum for the integral $J$ existed ("Dirichlet principle").

According to Riemann, potential theory as developed by Gauss and Dirichlet was well suited to a particular geometrical object, the "Riemann surface", he had introduced in order to study multi-valued functions such as algebraic functions and their integrals. Riemann required that the surface associated to a function be composed of as many sheets as were the branches of the function, connected in such a way to preserve continuity and to yield a single-valued function on the surface. In this way, he attained an abstract conception of the space of complex variables by means of a geometrical formulation which his contemporaries were to find very hard to understand. Referring to a conversation he had with Prym in 1874, Klein reported that Prym "*told me that* Riemann's surfaces originally are not necessarily many-sheeted surfaces over the plane, but that, on the contrary, complex functions of positions can be studied on arbitrarily given curved surfaces in exactly the same way as on the surfaces over the plane" [Klein 1882/1893, p. x].

Riemann made the surface simply connected with suitable transversal cuts (*Querschnitte*) and analysed the behaviour of the function in the neighbourhood of the singularities — poles and branch points. Then, thanks to the Dirichlet principle, Riemann stated and proved a fundamental existence theorem for a function with given singularities and boundary conditions [1851, p. 34–35]. This is the global theorem which, in Riemann's words, opens the way to the the study of complex functions independently of their analytical expressions [Riemann 1851, p. 35].

Many of the ideas of this paper, as Klein first emphasized, were inspired by physical topics. As Riemann told Betti (see [Bottazzini 1985, p. 559]), the idea of transversal cut on a surface struck him after a long discussion