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Entropy in operator algebras

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1 Introduction

While entropy has for a third of a century been a central concept in ergodic theory, its non-Abelian counterpart is still in its adolescent stage with only a few signs of mature strength. The signs, however, are promising and show a potential of a subject of importance in operator algebras, so much that I am glad to use this opportunity to take the reader on a guided tour of its ideas and their resulting definitions and theorems. I have also included some open problems with the hope that they may inspire further development of the subject into maturity. In addition to giving the necessary definitions I shall mainly be concerned with explicit formulas for entropy of automorphisms. I shall therefore not discuss entropy of endomorphisms and completely positive maps, nor will I say much about applications to physics.

There is another very promising approach to non-Abelian entropy which we shall not discuss but is presently persued by Voiculescu [32,33]. The definitions are quite different from the ones we shall give, but the values of the entropies are closely related to ours in nice cases, but are essentially different in general, see section 5.

2 Definitions and basic results

Before we embark on the non-Abelian definition of entropy let us recall the classical definition. We are then given a probability space $(X, \mathcal{B}, \mu)$ and a nonsingular measure preserving transformation T of X. If $\mathcal{P} = (P_1, \ldots, P_n)$ is a measurable partition of X we shall often identify it with the finite dimensional Abelian algebra generated by the characteristic functions $\mathcal{X}_{P_i}$. The entropy of $\mathcal{P}$ is

$$H(\mathcal{P}) = \sum_{i=1}^{n} \eta(\mu(P_i)),$$

where η is the real function on the unit interval, $\eta(t) = -t \log t$ for $t \in (0,1]$, and $\eta(0) = 0$. If $\mathcal{P} \vee \mathcal{Q}$ is the partition generated by two partitions $\mathcal{P}$ and $\mathcal{Q}$ then $H(\mathcal{P} \vee \mathcal{Q}) \leq H(\mathcal{P}) + H(\mathcal{Q})$, so we have convergence of the sequence

$$\frac{1}{k} H\left(\bigvee_{i=0}^{k-1} T^{-i}\mathcal{P}\right).$$

We denote by $H(T,\mathcal{P})$ its limit, and define the entropy of T by

$$H(T) = \sup_{\mathcal{P}} H(T,\mathcal{P}), \tag{2.1}$$

where the sup is taken over all finite measurable partitions. The crucial result for computing $H(T)$ is the Kolmogoroff-Sinai Theorem [34, 4.17].

Theorem 2.2. *If $\mathcal{P}$ is a generator, i.e. the σ-algebra generated by $(T^{-i}\mathcal{P})_{i\in\mathbb{Z}}$ equals $\mathcal{B}$, written $\bigvee_{-\infty}^{\infty} T^{-i}\mathcal{P} = \mathcal{B}$, then $H(T) = H(T,\mathcal{P})$.*

There is another version of this theorem which will be of interest in the sequel [34, 4.22].

Theorem 2.3. *If $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is an increasing sequence of partitions of X with $\bigvee_{n=1}^{\infty} \mathcal{P}_n = \mathcal{B}$ then*

$$H(T) = \lim_n H(T,\mathcal{P}_n)\,.$$

If one wants to extend the above definition of entropy to von Neumann algebras one is immediately confronted with a major obstacle. While there is a natural extension of the concept of finite partitions, namely finite dimensional von Neumann algebras, there is no natural candidate for the analogue of the partition $\mathcal{P}\vee\mathcal{Q}$ generated by $\mathcal{P}$ and $\mathcal{Q}$. Remember that the von Neumann algebra generated by two finite dimensional algebras can easily be infinite dimensional. However, if one considers the function $H\left(\bigvee_{i=1}^{k}\mathcal{P}_i\right)$ with $\mathcal{P}_i$ finite partitions as a function $H(\mathcal{P}_1,\ldots,\mathcal{P}_k)$ of k-variables, one can try to generalize this function. This will now be done following [8] for a von Neumann algebra M with a faithful normal finite trace τ such that $\tau(1) = 1$. In section 6 we shall see how this definition can be extended to general C^*-algebras and states.

For each $k\in\mathbb{N}$ denote by S_k the set of multiple indexed finite partitions of unity of M^+, $(x_{i_1\ldots i_k})_{i_j\in\mathbb{N}}$, i.e. each $x_{i_1\ldots i_k}\in M^+$, zero except for a finite number of indices and satisfying $\sum_{i_1,\ldots,i_k} x_{i_1\ldots i_k} = 1$.

For $x\in S_k$, $\ell\in\{1,\ldots,k\}$, $i_\ell\in\mathbb{N}$, we put

$$x^\ell_{i_\ell} = \sum_{i_1,\ldots,i_{\ell-1},i_{\ell+1},\ldots,i_k} x_{i_1\ldots i_k}\,.$$

If $N\subset M$ is a von Neumann subalgebra we denote by E_N the unique τ-invariant conditional expectation $E_N : M\to N$ defined by the identity

$$\tau(E_N(x)y) = \tau(xy), \qquad x\in M,\ y\in N\,.$$

Definition 2.4. Let $N_1, \dots, N_k$ be finite dimensional von Neumann subalgebras of M. Then

$$H(N_1, \dots, N_k) = \sup_{x \in S_k} \left\{ \sum_{i_1, \dots, i_k} \eta(\tau(x_{i_1 \dots i_k})) - \sum_{\ell=1}^{k} \sum_{i_\ell} \tau(\eta(E_{N_\ell}(x^\ell_{i_\ell}))) \right\}.$$

Since the trivial partition $x = (1)$ gives the value zero, $H \geq 0$. Also it is clear that H is symmetric in the N's. Furthermore H satisfies the following nice requirements

(A) $H(N_1, \dots, N_k) \leq H(P_1, \dots, P_k)$ when $N_j \subset P_j$, $j = 1, \dots, k$.

(B) $H(N_1, \dots, N_k) \leq H(N_1, \dots, N_j) + H(N_{j+1}, \dots, N_k)$ for $1 \leq j < k$.

(C) $N_1, \dots, N_j \subset N \Rightarrow H(N_1, \dots, N_j, N_{j+1}, \dots, N_k) \leq H(N, N_{j+1}, \dots, N_k)$.

(D) For any family of minimal projections of N, $(e_\alpha)_{\alpha \in I}$ such that $\sum_{\alpha \in I} e_\alpha = 1$ one has $H(N) = \sum_{\alpha \in I} \eta\tau(e_\alpha)$.

(E) If $(N_1 \cup \cdots \cup N_k)''$ is generated by pairwise commuting von Neumann subalgebras P_j of N_j then

$$H(N_1, \dots, N_k) = H\big((N_1 \cup \cdots \cup N_k)''\big).$$

The crucial technical ingredient in the proof of the above properties, and in particular of (C), is the relative entropy of two states, or rather positive operators in our case, defined by

$$S(x|y) = \tau(x(\log x - \log y)), \qquad x, y \in M^+,\ x \leq \lambda y$$

for some $\lambda > 0$. For general normal states of von Neumann algebras the relative entropy is defined by Araki [1] via the relative modular operator of the two states, and by Pusz and Woronowicz [25] for states of C^*-algebras. The main property of S is that it is a jointly convex function in x and y [16], see also [15] and [25].

Having H it is now an easy matter to extend the classical definition (2.1) of entropy. We look at the measure preserving transformation T on $(X, \mathcal{B}, \mu)$ as an automorphism α_T of the Abelian von Neumann algebra $L^\infty(X, \mathcal{B}, \mu)$ defined by $\alpha_T(f) = f \circ T^{-1}$, and partitions as finite dimensional algebras.

Definition 2.5 Let α be an automorphism of M such that $\tau \circ \alpha = \tau$. If $N \subset M$ is finite dimensional we let

$$H(\alpha, N) = \lim_{k \to \infty} \tfrac{1}{k} H(N, \alpha(N), \dots, \alpha^{k-1}(N)),$$

where as in the classical case the sequence converges by the subadditivity of H, property (B). The *entropy* of α is

$$H(\alpha) = \sup_N H(\alpha, N),$$

where the sup is taken over all finite dimensional subalgebras $N \subset M$.

Remark 2.6. If $P \subset M$ is a von Neumann subalgebra such that $\alpha(P) = P$, it is immediate from the definition that the restriction $\alpha|P$ satisfies $H(\alpha|P) \leq H(\alpha)$.

Remark 2.7. If α is periodic then $H(\alpha, N) = 0$ for all N, hence $H(\alpha) = 0$. More generally, if α is contained in a compact subgroup of $\mathrm{Aut}(M)$ then Besson [2] has shown that we still have $H(\alpha) = 0$.

To compute $H(\alpha)$ it is as in the classical case necessary to reduce the choice of N's. The following concept is helpful for this purpose.

Definition 2.8. If N and P are finite dimensional von Neumann subalgebras of M their *relative entropy* is

$$H(N|P) = \sup_{x \in S_1} \sum_i (\tau\eta E_P(x_i) - \tau\eta E_N(x_i)).$$

$H(N|P)$ has the following nice properties:

(F) $H(N_1, \ldots, N_k) \leq H(P_1, \ldots, P_k) + \sum_{j=1}^{k} H(N_j|P_j)$.

(G) $H(N|Q) \leq H(N|P) + H(P|Q)$.

(H) $H(N|P)$ is increasing in N and decreasing in P.

(I) If N and P commute then

$$H(N|P) = H((N \cup P)''|P) = H(H \cup P)'') - H(P).$$

Properties (F), (G), (H) are easy to prove, while (I) is a consequence of the Lieb-Ruskai second strong subadditivity property [17]. The relative entropy is continuous in the following sense.

Theorem 2.9. *For all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all pairs of von Neumann subalgebras N and P of M with $\dim N = n$, we have*

$$N \overset{\delta}{\subset} P \Rightarrow H(N|P) < \varepsilon\,.$$

Here $N \overset{\delta}{\subset} P$ means that for all $x \in N$, $\|x\| \leq 1$, there exists $y \in P$ with $\|y\| \leq 1$ such that $\|x - y\|_2 < \delta$, where $\|z\|_2 = \tau(z^*z)^{1/2}$. This result together with property (F) is very useful in restricting the choice of N in the definition of $H(\alpha)$. An example is the proof of the generalization of the Kolmogoroff-Sinai Theorem (2.3).

Theorem 2.10. *Suppose M is hyperfinite with an increasing sequence $(P_n)_{n \in \mathbb{N}}$ of finite dimensional subalgebras with union weakly dense in M. Then if $\alpha \in \mathrm{Aut}\,(M)$ and $\tau \circ \alpha = \tau$ we have*

$$H(\alpha) = \lim_n H(\alpha, P_n)\,.$$