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UNIFORMIZING GROMOV HYPERBOLIC SPACES

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Dedicated to Jussi Väisälä

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Abstract. — The unit disk in the complex plane has two conformally related lives: one as an incomplete space with the metric inherited from $\mathbb{R}^2$, the other as a complete Riemannian 2-manifold of constant negative curvature. Consequently, problems in conformal analysis can often be formulated in two equivalent ways depending on which metric one chooses to use. The purpose of this volume is to show that a similar choice is available in much more generality. We shall replace the incomplete disk by a uniform metric space (defined as a generalization of a uniform domain in $\mathbb{R}^n$) and the space of constant negative curvature by a general Gromov hyperbolic space. We then prove that there is a one-to-one correspondence between quasiisometry classes of (proper, geodesic, and roughly starlike) Gromov hyperbolic spaces and the quasisimilarity classes of bounded locally compact uniform spaces. We study Euclidean domains that are Gromov hyperbolic with respect to the quasihyperbolic metric and the Martin boundaries of such domains. A characterization of planar Gromov hyperbolic domains is given. We also study quasiconformal homeomorphisms of Gromov hyperbolic spaces of bounded geometry; under mild conditions on the spaces we prove that such maps are rough quasiisometries. We employ a version of the classical Gehring-Hayman theorem, and methods from analysis on metric spaces such as modulus estimates on Loewner spaces.

Résumé **(Uniformisation des espaces hyperboliques de Gromov)**

On peut considérer le disque unité dans le plan complexe de deux façons différentes : comme un espace incomplet si on le munit de la métrique euclidienne de $\mathbb{R}^2$, et comme un espace complet s'il est équipé d'une métrique de courbure négative constante. Par conséquent, on peut souvent formuler des problèmes d'analyse conforme de deux manières différentes, suivant la métrique que l'on choisit d'utiliser. L'objet de ce volume est de montrer qu'un choix semblable est possible de manière beaucoup plus générale. On remplace le disque incomplet par un espace uniforme (défini comme une généralisation d'un domaine uniforme dans $\mathbb{R}^n$) et l'espace de courbure négative constante par un espace hyperbolique au sens de Gromov. On montre ensuite qu'il y a une correspondance univoque entre les classes de quasi-isométrie des espaces hyperboliques (qui sont de plus propres, géodésiques et grossièrement étoilés) et les classes de quasi-similitudes des espaces uniformes qui sont bornés et localement compacts. Nous étudions les domaines euclidiens munis de la métrique quasi-hyperbolique qui sont hyperboliques au sens de Gromov, et les frontières de Martin de ces domaines. On donne une caractérisation de domaines hyperboliques dans le plan. Nous étudions aussi les homéomorphismes quasi-conformes entre des espaces hyperboliques qui satisfont à une condition de géométrie bornée ; sous des hypothèses modérées, on démontre que les applications comme ci-dessus sont des quasi-isométries au sens large. Nous utilisons une version du théorème classique de Gehring-Hayman, et des méthodes d'analyse sur les espaces métriques comme des estimations de module dans les espaces de Loewner.

CONTENTS

CHAPTER 1

INTRODUCTION

The unit disk in the complex plane has two conformally related lives: one as an incomplete space with the metric inherited from $\mathbb{R}^2$, the other as a complete Riemannian 2-manifold of constant negative curvature. Consequently, problems in conformal analysis often can be formulated in two equivalent ways, depending on which metric one chooses to use. The purpose of this paper is to show that a similar choice is available in much more generality. We shall replace the incomplete disk by a uniform metric space (to be defined below as a generalization of a uniform domain in $\mathbb{R}^n$) and the space of constant negative curvature by a general Gromov hyperbolic metric space. We then have the following theorem:

Theorem 1.1. — *There is a one-to-one (conformal) correspondence between the quasiisometry classes of proper geodesic roughly starlike Gromov hyperbolic spaces and the quasisimilarity classes of bounded locally compact uniform spaces.*

The terminology of this introduction will be explained in the course of the paper. The proof of Theorem 1.1 is given in Chapter 4. It relies heavily on the ideas around the Gehring-Hayman theorem, which is discussed in Chapter 5. (We believe that our study of the Gehring-Hayman theorem in Chapter 5 has independent merit.) In subsequent chapters, we shall give several applications, old and new, of Theorem 1.1. Besides the Gehring-Hayman theorem, the paper contains other related studies of independent interest, as will be discussed later in this introduction.

A metric space is called *proper* if its closed balls are compact; it is called *geodesic* if every pair of points in it can be joined by a *geodesic*, that is, by a curve whose length equals the distance between the points. A geodesic metric space is *Gromov hyperbolic* if every geodesic triangle in it is *δ-thin* for some fixed $\delta \geqslant 0$. We also use the term *δ-hyperbolic* in this case. See **[Gr2]**, **[GhHa**, pages 16, 41, and 60], and Chapter 3 below for precise definitions.