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Thomas Duquesne, Jean-François Le Gall

Abstract. — We investigate the genealogical structure of general critical or subcritical continuous-state branching processes. Analogously to the coding of a discrete tree by its contour function, this genealogical structure is coded by a real-valued stochastic process called the height process, which is itself constructed as a local time functional of a Lévy process with no negative jumps. We present a detailed study of the height process and of an associated measure-valued process called the exploration process, which plays a key role in most applications. Under suitable assumptions, we prove that whenever a sequence of rescaled Galton-Watson processes converges in distribution, their genealogies also converge to the continuous branching structure coded by the appropriate height process. We apply this invariance principle to various asymptotics for Galton-Watson trees. We then use the duality properties of the exploration process to compute explicitly the distribution of the reduced tree associated with Poissonnian marks in the height process, and the finite-dimensional marginals of the so-called stable continuous tree. This last calculation generalizes to the stable case a result of Aldous for the Brownian continuum random tree. Finally, we combine the genealogical structure with an independent spatial motion to develop a new approach to superprocesses with a general branching mechanism. In this setting, we derive certain explicit distributions, such as the law of the spatial reduced tree in a domain, consisting of the collection of all historical paths that hit the boundary.

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Résumé **(Arbres aléatoires, processus de Lévy et processus de branchement spatiaux)**
Nous étudions la structure généalogique de processus de branchement critiques ou sous-critiques à espace d'états continu. De manière analogue au codage d'un arbre discret par son contour, cette structure généalogique est codée par un processus aléatoire appelé le processus des hauteurs, qui est lui-même construit comme une fonctionnelle de type temps local d'un processus de Lévy sans saut négatif. Nous présentons une étude détaillée du processus des hauteurs et d'un processus à valeurs mesures associé appelé le processus d'exploration. Sous des hypothèses convenables, nous montrons que si une suite de processus de Galton-Watson convenablement changés d'échelle converge en loi, leurs généalogies convergent aussi vers la structure de branchement codée par le processus des hauteurs. Nous appliquons ce principe d'invariance à divers théorèmes limites pour les arbres de Galton-Watson. A l'aide des propriétés de dualité du processus d'exploration, nous calculons la loi de l'arbre réduit associé à des marques poissonniennes dans le processus des hauteurs, et les lois marginales de dimension finie de l'arbre continu stable. Ce dernier calcul généralise au cas stable un résultat d'Aldous pour l'arbre brownien continu. Finalement, en combinant la structure généalogique avec un déplacement spatial, nous développons une nouvelle approche des superprocessus avec un mécanisme de branchement général. Dans ce cadre, nous obtenons certaines distributions explicites, dont celle de l'arbre spatial réduit dans un domaine, qui décrit toutes les trajectoires historiques ayant atteint la frontière.

CONTENTS

INTRODUCTION

The main goal of this work is to investigate the genealogical structure of continuous-state branching processes in connection with limit theorems for discrete Galton-Watson trees. Applications are also given to the construction and various properties of spatial branching processes including a general class of superprocesses.

Our starting point is the recent work of Le Gall and Le Jan [**32**] who proposed a coding of the genealogy of general continuous-state branching processes via a real-valued random process called the height process. Recall that continuous-state branching processes are the continuous analogues of discrete Galton-Watson branching processes, and that the law of any such process is characterized by a real function ψ called the branching mechanism. Roughly speaking, the height process is a continuous analogue of the contour process of a discrete branching tree, which is easy to visualize (see Section 0.1, and note that the previous informal interpretation of the height process is made mathematically precise by the results of Chapter 2). In the important special case of the Feller branching diffusion ($\psi(u) = u^2$), the height process is reflected linear Brownian motion: This unexpected connection between branching processes and Brownian motion, or random walk in a discrete setting has been known for long and exploited by a number of authors (see e.g. [**3**], [**11**], [**18**], [**39**], [**42**]). The key contribution of [**32**] was to observe that for a general subcritical continuous-state branching process, there is an explicit formula expressing the height process as a functional of a spectrally positive Lévy process whose Laplace exponent ψ is precisely the branching mechanism. This suggests that many problems concerning the genealogy of continuous-state branching processes can be restated and solved in terms of spectrally positive Lévy processes, for which a lot of information is available (see e.g. Bertoin's recent monograph [**5**]). It is the principal aim of the present work to develop such applications.

In the first two sections below, we briefly describe the objects of interest in a discrete setting. In the next sections, we outline the main contributions of the present work.

0.1. Discrete trees

Let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. If $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, we set $|u| = n$, so that $|u|$ represents the "generation" of u. If $u = (u_1, \ldots u_m)$ and $v = (v_1, \ldots, v_n)$ belong to $\mathcal{U}$, we write $uv = (u_1, \ldots u_m, v_1, \ldots, v_n)$ for the concatenation of u and v. In particular $u\varnothing = \varnothing u = u$.

A (finite) rooted ordered tree $\mathcal{T}$ is a finite subset of $\mathcal{U}$ such that:

(i) $\varnothing \in \mathcal{T}$.

(ii) If $v \in \mathcal{T}$ and $v = uj$ for some $u \in \mathcal{U}$ and $j \in \mathbb{N}$, then $u \in \mathcal{T}$.

(iii) For every $u \in \mathcal{T}$, there exists a number $k_u(\mathcal{T}) \geqslant 0$ such that $uj \in \mathcal{T}$ if and only if $1 \leqslant j \leqslant k_u(\mathcal{T})$.

We denote by $\mathbf{T}$ the set of all rooted ordered trees. In what follows, we see each vertex of the tree $\mathcal{T}$ as an individual of a population whose $\mathcal{T}$ is the family tree. The cardinality $\#(\mathcal{T})$ of $\mathcal{T}$ is the total progeny.

If $\mathcal{T}$ is a tree and $u \in \mathcal{T}$, we define the shift of $\mathcal{T}$ at u by $\theta_u \mathcal{T} = \{v \in U : uv \in \mathcal{T}\}$. Note that $\theta_u \tau \in \mathbf{T}$.

We now introduce the (discrete) *height function* associated with a tree $\mathcal{T}$. Let us denote by $u(0) = \varnothing, u(1), u(2), \ldots, u(\#(\mathcal{T}) - 1)$ the elements of $\mathcal{T}$ listed in lexicographical order. The height function $H(\mathcal{T}) = (H_n(\mathcal{T}); 0 \leqslant n < \#(\mathcal{T}))$ is defined by

$$H_n(\mathcal{T}) = |u(n)|, \quad 0 \leqslant n < \#(\mathcal{T}).$$

The height function is thus the sequence of the generations of the individuals of $\mathcal{T}$, when these individuals are visited in the lexicographical order (see Fig. 1 for an example). It is easy to check that $H(\mathcal{T})$ characterizes the tree $\mathcal{T}$.

The *contour function* gives another way of characterizing the tree, which is easier to visualize on a picture (see Fig. 1). Suppose that the tree is embedded in the half-plane in such a way that edges have length one. Informally, we imagine the motion of a particle that starts at time $t = 0$ from the root of the tree and then explores the tree from the left to the right, moving continuously along the edges at unit speed, until it comes back to its starting point. Since it is clear that each edge will be crossed twice in this evolution, the total time needed to explore the tree is $\zeta(\mathcal{T}) := 2(\#(\mathcal{T}) - 1)$. The value C_t of the contour function at time t is the distance (on the tree) between the position of the particle at time t and the root. By convention $C_t = 0$ if $t \geqslant \zeta(\mathcal{T})$. Fig. 1 explains the definition of the contour function better than a formal definition.

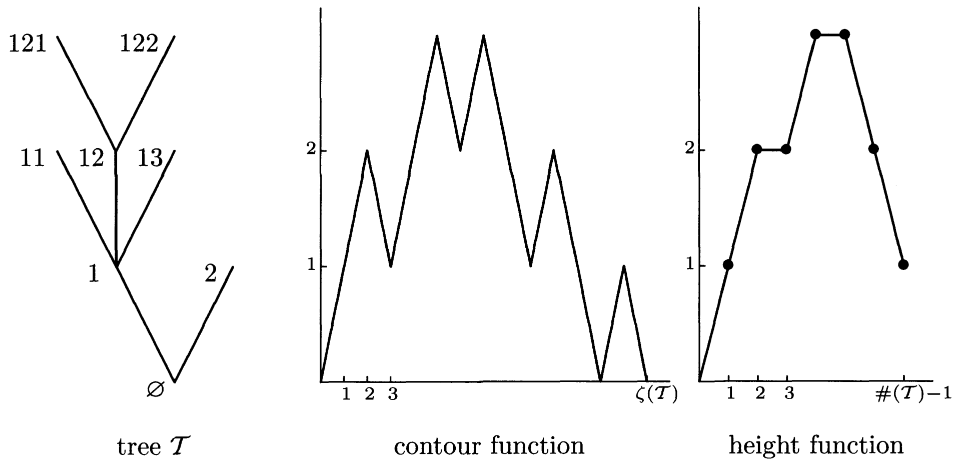

FIGURE 1

0.2. Galton-Watson trees

Let μ be a critical or subcritical offspring distribution. This means that μ is a probability measure on $\mathbb{Z}_+$ such that

$$\sum_{k=0}^{\infty} k\mu(k) \leqslant 1.$$

We exclude the trivial case where $\mu(1) = 1$.

There is a unique probability distribution $\mathbf{Q}_\mu$ on $\mathbf{T}$ such that

(i) $\mathbf{Q}_\mu(k_\varnothing = j) = \mu(j), \quad j \in \mathbb{Z}_+$.

(ii) For every $j \geqslant 1$ with $\mu(j) > 0$, the shifted trees $\theta_1\mathcal{T}, \ldots, \theta_j\mathcal{T}$ are independent under the conditional probability $\mathbf{Q}_\mu(\cdot \mid k_\varnothing = j)$ and their conditional distribution is $\mathbf{Q}_\mu$.

A random tree with distribution $\mathbf{Q}_\mu$ is called a Galton-Watson tree with offspring distribution μ, or in short a μ-Galton-Watson tree.

Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be a sequence of independent μ-Galton-Watson trees. We can associate with this sequence a *height process* obtained by concatenating the height functions of each of the trees $\mathcal{T}_1, \mathcal{T}_2, \ldots$. More precisely, for every $k \geqslant 1$, we set

$$H_n = H_{n-(\#(\mathcal{T}_1)+\cdots+\#(\mathcal{T}_{k-1}))}(\mathcal{T}_k) \text{ if } \#(\mathcal{T}_1)+\cdots+\#(\mathcal{T}_{k-1}) \leqslant n < \#(\mathcal{T}_1)+\cdots+\#(\mathcal{T}_k).$$

The process $(H_n, n \geqslant 0)$ codes the sequence of trees.

Similarly, we define a *contour process* $(C_t, t \geqslant 0)$ coding the sequence of trees by concatenating the contour functions

$$(C_t(\mathcal{T}_1), t \in [0, \zeta(\mathcal{T}_1) + 2]), \ (C_t(\mathcal{T}_2), t \in [0, \zeta(\mathcal{T}_2) + 2]), \text{ etc.}$$