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ORBIT CLOSURES IN FLAG VARIETIES FOR THE CENTRALIZER OF AN ORDER-TWO NILPOTENT ELEMENT: NORMALITY AND RESOLUTIONS FOR TYPES A, B, D

by Simon Jacques

Abstract. — Let G be a reductive algebraic group in classical types A, B, D. Let e be an element of the Lie algebra of G, with $Z \subset G$ its centralizer for the adjoint action. We assume that e identifies with a nilpotent matrix of order two, which guarantees that the number of Z-orbits in the flag variety of G is finite. For types B and D in characteristic two, we also assume that the image of e is totally isotropic. We show that the closure Y of such an orbit is normal. We also prove that Y is Cohen-Macaulay with rational singularities provided that the base field is of characteristic zero, and that Cohen–Macaulayness holds in any characteristic for type A. We exhibit a rational and birational morphism onto Y involving Schubert varieties. Our work generalizes a result by N. Perrin and E. Smirnov on the Springer fibers.

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Résumé (*Adhérences de certaines orbites dans la variété de drapeaux, résolution et normalité dans les types classiques A, B, D*). — Soit G un groupe algébrique réductif en type A, B ou D. Soit e un élément de l'algèbre de Lie de G et $Z \subset G$ son centralisateur, agissant sur la variété de drapeaux G/B de G. Nous supposons que e s'identifie à une matrice nilpotente d'ordre deux, ce qui garantit un nombre fini de Z-orbites dans G/B. Pour les types B et D en caractéristique deux, nous supposons également que l'image de e est totalement isotrope. Nous montrons alors que toute adhérence Y de Z-orbite dans G/B est normale. Nous prouvons également que Y est de Cohen-Macaulay avec des singularités rationnelles sous l'hypothèse que la caractéristique du corps de base est zéro, et que cette propriété de Cohen-Macaulay est vraie en toute caractéristique pour le type A. Pour cela, nous construisons un morphisme rationnel et birationnel sur Y au moyen de variétés de Schubert. Notre travail généralise un résultat de N. Perrin et E. Smirnov sur les fibres de Springer.

Introduction

1. — Let $\Bbbk$ be an algebraically closed field and let G be a reductive connected algebraic group over $\Bbbk$, with B a Borel subgroup. Let e be a nilpotent element of the Lie algebra $\mathfrak{g}$ of G, and let Z be its centralizer in G for the adjoint action. When the number of Z-orbits in the flag variety G/B is finite, their closures are of particular interest. They include, in this case, the irreducible components of the so-called Springer fiber over e. It is the fiber of e under the proper birational morphism

$$\widetilde{\mathcal{N}} \to \mathcal{N},$$

called the Springer resolution, which is the projection onto the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ from the smooth variety $\widetilde{\mathcal{N}} := \{\, (x, gB) \in \mathcal{N} \times G/B \mid Ad\ g^{-1} \cdot x \in \mathfrak{b} \,\}$, $\mathfrak{b}$ denoting the Lie algebra of B. The Springer fibers are of main interest in representation theory (see the seminal work of T.A. Springer [35], their link with the orbital varieties [34] and the Steinberg variety [38][1]). They are connected and equidimensional (see, for example, [34]), and their irreducible components have been the subject of numerous studies. For the classical cases and char($\Bbbk$) $\neq$ 2, N. Spaltenstein (type A, [34]) and M. van Leeuwen (types B, C, D, [26]) showed that they are parameterized by standard and domino tableaux, whose shapes are given by Young diagrams relative to the nilpotent

1. Actually, the latter references deal with unipotent elements instead of nilpotent ones, regarding the Springer fibers as the variety of Borel subgroups containing a given unipotent element. However, recall that when G is the general linear group or is almost simple and simply connected, and the characteristic of $\Bbbk$ is good, the unipotent variety in the group G and the nilpotent cone in its Lie algebra $\mathfrak{g}$ can be identified with a G-equivariant isomorphism (see, for example, [35, Theorem 3.1] for an original but weaker statement and [20, Theorem 6.20] and [1, Corollary 9.3.3] for this more general one), so that the two notions of Springer fibers match exactly.

orbit in question. Subsequent studies of their singularities have often been based on these shapes and have mainly produced results for G the general linear group and $\Bbbk$ the field of complex numbers. For an example, F. Fung showed in [14] that they are all smooth in the so-called hook and two-line cases. A. Melnikov and L. Fresse gave a necessary and sufficient condition for this global smoothness in [12], while they gave a criterion for individual smoothness in [11, 12], under the additional assumption of being in the *two column case*. This is the first case where singularities appear. It also implies that the order of nilpotency of $ad\ e$ is less than or equal to 3, which is a condition ensuring, after the work of Panyushev [30], that the number of Z-orbits is finite (in fact, this implication is established for $\mathrm{char}(\Bbbk) = 0$, but for our types A, B, D, it is still valid for the other characteristics; see Propositions 2.3 and 2.9).

2. — The *two-column case* is assumed in the article [31] by N. Perrin and E. Smirnov. For type A and $\mathrm{char}(\Bbbk) \neq 2$, they present rational resolutions of the components and show that they are normal and Cohen–Macaulay. They also give arguments for the same results in type D, but there is a gap in their proof of normality and the Cohen–Macaulay property, due to the nonalgebraicity of a certain map (see Appendix B for details and a counterexample). Nevertheless, their proof of the existence of a rational birational morphism onto the component is still valid for this type. Our work is mainly inspired by the latter, generalizing it in several directions. Retaining the assumption of the two-column case, we also prove normality and rationality, but for the much broader class of Z-orbit closures. For example, if $G = Gl_{n\Bbbk}$ is the general linear group, and r represents the rank of e considered as a nilpotent matrix of order two, then we can deduce from our Proposition 2.3, and the hook-length formula that the number of Z-orbits is $(n-r+1)(n-r)\dots(n-2r+2)$ times the number of irreducible components. In addition, we consider the three types A, B, D and we also deal with the case $\mathrm{char}(\Bbbk) = 2$ (with precautions regarding the nilpotent orbit considered; see below).

3. — Let us now state our main results. We assume that $\Bbbk$ is of arbitrary characteristic and we fix an integer n. Let $O_{n\Bbbk}$ be the group over $\Bbbk$ whose closed points are the invertible $n \times n$ matrices preserving the quadratic form

$$\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} Y_k Y_{n-k+1}. \tag{1}$$

For even n, let us denote by $\triangle_n$ the Dickson invariant, as defined, for example, in [24, IV, §5]. This is the regular function on $O_{n\Bbbk}$ satisfying $det_n = 1 + 2\triangle_n$, where det_n is the restriction of the determinant to $O_{n\Bbbk}$ (see Section 1.3 for details). We then define the special orthogonal group $SO_{n\Bbbk}$ as the zero locus of $\triangle_n$ if n is even and as that of $det_n - 1$ if n is odd. Without these precautions, note that it fails to be connected and semi-simple of type B_n (odd n) or D_n (even n) in the case $\mathrm{char}(\mathbb{k}) = 2$ (see, for example, [18], [8, Appendix C] and Section 1.4).

Assume now that G is the general linear group $Gl_{n\mathbb{k}}$ or the special orthogonal group $SO_{n\mathbb{k}}$. We also assume that the nilpotent element e is identified with a nilpotent matrix of order two, which means that we are in the two-column case. If $\mathrm{char}(\mathbb{k}) = 2$ and $G = SO_{n\mathbb{k}}$, we make the additional assumption that the image of e is totally isotropic.

Recall that a proper morphism $f : X \to Y$ of locally noetherian schemes is called *rational* if $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$ and $R^i f_*\mathcal{O}_X = 0$ for $i > 0$. When the schemes are irreducible varieties with X smooth, such a rational morphism f is said to be a *rational resolution* if it is also birational with $R^i f_*\omega_X = 0$ for $i > 0$, where ω_X denotes the canonical bundle of X. If $\mathrm{char}(\mathbb{k}) = 0$, two rational resolutions can be dominated by a third, so being the target of a rational resolution leads to the intrinsic notion of *having rational singularities.* We prove the following.

Theorem 0.1. — *The Z-orbit closures in the flag variety of G are normal. In characteristic zero, they are Cohen–Macaulay with rational singularities. In any other characteristic, they remain Cohen–Macaulay in type A.*

This theorem is based on two results. The first is the construction of an explicit birational morphism using matrix models and involving Schubert varieties. It ensures the existence of a Borel subgroup B of G, containing a maximal torus T, and of a closed reductive subgroup H of G equipped with a retraction $\varpi : Z \to H$, having $B_H := B \cap H$ as a Borel subgroup and $T_H := T \cap H$ as the maximal torus, so that we have the following.

Theorem 0.2. — *For any Z-orbit closure Y in G/B, there exists w in the Weyl group of G such that $Y = \overline{HB \cdot wB} = \overline{Z \cdot wB}$ and*

$$H \times^{B_H} \overline{B \cdot wB} \to Y,\ \ [h, gB] \mapsto hgB \tag{2}$$

is rational, birational, Z-equivariant, with a Z-action on $H \times^{B_H} \overline{B \cdot wB}$ defined by $z \cdot [h, gB] = [\varpi(z)h, h^{-1}\varpi(z)^{-1}zhgB]$.

The second result is valid in a more general context, where we assume only that G is a connected reductive group over $\mathbb{k}$, and H a closed connected reductive subgroup of G. We make the same assumptions as before about T, B, T_H, B_H and fix any w in the Weyl group of G. We denote by ρ_G the half-sum of positive roots and, for any dominant character λ, by $V_G(\lambda)$ the dual Weyl G-module with lowest weight $-\lambda$. Let ρ_H and $V_H(\lambda)$ also be the corresponding objects for H. We refer to Section 3 for details of the notation and a stronger result that also deals with the vanishing of the canonical bundle.

Theorem 0.3. — *Let us assume the following.*

(i) *The morphism* $\pi\colon H \times^{B_H} \overline{B \cdot wB} \to \overline{HB \cdot wB}, [h, gB] \mapsto hgB$ *is birational.*
(ii) *The character* $2\rho_H - \rho_{G|T_H}$ *is dominant.*
(iii) $\mathrm{char}(\Bbbk) = 0$ *or*
(iii)′ $\mathrm{char}(\Bbbk) = p > 0$ *and the restriction* $V_G((p-1)\rho_G) \to V_H((p-1)\rho_{G|T_H})$ *is surjective.*

Then $\overline{HB \cdot wB}$ *is normal and* π *is rational.*

Remark 0.4. — It remains an open question whether the Cohen–Macaulay property is valid for types B and D and whether this property, rationality and normality are valid for type C and the exceptional types.

Remark 0.5. — If we take $H = T$ in Theorem 0.3, we find the well-known result on the normality of Schubert varieties. In fact, in this case, the sequence of arguments used in the proof coincides with that of M. Brion and S. Kumar in [4].

Remark 0.6. — In types B and D, if $\mathrm{char}(\Bbbk) \neq 2$, the assumption of being in the two-column case implies that the image of e is totally isotropic. However, this is not the case if $\mathrm{char}(\Bbbk) = 2$, as can be seen in type D by taking $e := \left(\begin{smallmatrix} 0&1&1&0\\ 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0 \end{smallmatrix}\right)$ which is a matrix of nilpotency order two as $e' := \left(\begin{smallmatrix} 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{smallmatrix}\right)$. Note that the dimensions of the centralizers of e and e' differ (they are 2 and 4, respectively, see [18, Theorem 4.5]), so we can see that the rank of nilpotent elements does not suffice to characterize nilpotent orbits in this case. Our assumption about the image of e is, therefore, necessary to work in our matrix model and then apply our reasoning (which crucially depends on the dimension of Z). It also turns out to be sufficient (see Section 2.2 and the result of [18, Theorem 3.8]).

Remark 0.7. — Let us assume that $\mathrm{char}(\Bbbk) \neq 2$, that the rank of e is odd and that the Z-orbit in question has a T-fixed point (which is not superfluous, since there are orbits without such points; see Proposition 2.9). Finally, let us replace Z by its neutral component Z^0. Theorem 0.2 (with the exception of rationality) is then again valid for $G = Sp_{n\Bbbk}$ the symplectic group over $\Bbbk$ (see the matrix models in Section 1 for a concrete description of this group). However, our proof of Theorem 0.1 cannot work because of the nondominance of the character involved in Theorem 0.3 (see Remark 4.2).

4. — To prove these results, we take up many of Perrin and Smirnov's arguments in [31] while developing new techniques. Our birational morphism (2) onto a Z-orbit closure is quite analogous to Perrin and Smirnov's morphism targeting an irreducible component of the Springer fiber (see (36) in Appen-