Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

SKT HYPERBOLIC AND GAUDUCHON HYPERBOLIC COMPACT COMPLEX MANIFOLDS

Samir Marouani

Tome 152
Fascicule 3

2024

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 519-545

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 152, septembre 2024

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 160 € ($ 240),
avec supplément papier : Europe 244 €, hors Europe 330 € ($ 421)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
152 (3), 2024, p. 519-545

SKT HYPERBOLIC AND GAUDUCHON HYPERBOLIC COMPACT COMPLEX MANIFOLDS

by Samir Marouani

Abstract. — We introduce two notions of hyperbolicity for not necessarily Kähler even balanced n-dimensional compact complex manifolds X. The first, called *SKT hyperbolicity*, generalises Gromov's Kähler hyperbolicity by means of SKT metrics. The second, called *Gauduchon hyperbolicity* by means of Gauduchon metrics. Our first main result in this paper asserts that every SKT hyperbolic X is also Kobayashi/Brody hyperbolic and every Gauduchon hyperbolic X is divisorially hyperbolic. The second main result is to prove a vanishing theorem for the L^2 harmonic spaces on the universal cover of an SKT hyperbolic manifold.

Résumé (*Hyperbolicité SKT et hyperbolicité de Gauduchon pour les variétés complexes compactes*). — Nous introduisons deux notions d'hyperbolicité pour les variétés complexes compactes X de dimension n, non nécessairement kählériennes ni équilibrées. La première, appelée hyperbolicité SKT, généralise l'hyperbolicité kählérienne au sens de Gromov à l'aide d'une métrique SKT. La seconde, appelée hyperbolicité de Gauduchon. Notre premier résultat principal dans cet article affirme que toute variété SKT hyperbolique est également hyperbolique au sens de Kobayashi/Brody, et toute variété Gauduchon hyperbolique est divisoriellement hyperbolique. Notre deuxième résultat principal démontre un théorème d'annulation pour les espaces harmoniques L^2 sur le revêtement universel d'une variété SKT hyperbolique.

Texte reçu le 20 mai 2023, modifié le 28 janvier 2024, accepté le 22 février 2024.

Samir Marouani, Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France • *E-mail :* almarouanisamir@gmail.com

Mathematical subject classification (2010). — 32Q45, 53C55, 14F40.

Key words and phrases. — Gauduchon Hermitian metrics, SKT Hermitian metrics, hyperbolic compact complex manifolds.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2024/519/$ 5.00
doi:10.24033/bsmf.2894

1. Introduction

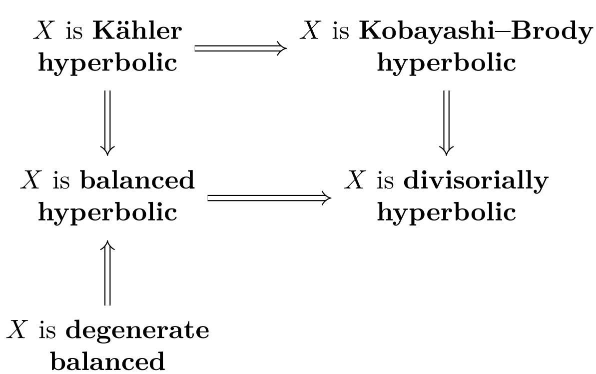
S. Kobayashi called a complex manifold X, which need not be either Kähler or compact, *hyperbolic* if the pseudo-distance he had introduced on X is actually a distance. Using the mapping decreasing property of this distance, one can show that every holomorphic map from the complex plane $\mathbb{C}$ to a *Kobayashi hyperbolic* manifold is *constant.* Conversely, Brody observed that a compact complex manifold X is Kobayashi hyperbolic if every holomorphic map from $\mathbb{C}$ to X is constant. The long-standing Kobayashi–Lang conjecture predicts that, for a compact Kähler manifold X, if X is Kobayashi hyperbolic then its canonical bundle K_X is *ample.*

M. Gromov introduced in one of his seminal papers [7], the notion of *Kähler hyperbolicity* for a compact Kähler manifold X. The manifold X is called *Kähler hyperbolic* if X admits a Kähler metric ω whose lift $\widetilde{\omega}$ to the universal cover $\tilde{X}$ of X can be expressed as

$$\widetilde{\omega} = d\alpha$$

for a *bounded* 1-form α on $\tilde{X}$. As pointed out by Gromov, it is not hard to see that the Kähler hyperbolicity implies the Kobayashi hyperbolicity.

The Kähler hyperbolicity is generalized in [10] to what we call **balanced hyperbolicity**. This is done by replacing the Kähler metric in the Kähler hyperbolicity by a *balanced metric.* Meanwhile, a compact complex n-dimensional manifold X is said to be balanced hyperbolic if it carries a balanced metric ω such that ω^{n-1} is $\widetilde{d}$-bounded. The Brody hyperbolicity is replaced by what we call **divisorial hyperbolicity**. A compact complex manifold X is called *divisorially hyperbolic* if there exists no non-trivial holomorphic map from $\mathbb{C}^{n-1}$ to X satisfying certain *subexponential volume growth conditions.* Where the main result in [10] asserts that every balanced hyperbolic X is also divisorially hyperbolic (Thm 2.8), and therefore the following implication holds:



In this paper, we present a hyperbolicity theory where the Kähler metric is replaced by the SKT metric and the balanced metric is replaced by the Gauduchon metric on n-dimensional compact complex manifolds. The notions we introduce are weaker, thus more inclusive, than their classical counterparts. In particular, the setting does not need to be Kähler or even balanced. Our motivation stems from the existence of many interesting examples of non-Kähler compact complex manifolds that exhibit hyperbolicity features in a generalized sense, which we now set out to explain. Our first main result in this paper asserts that every SKT hyperbolic X is also Kobayashi hyperbolic, and every Gauduchon hyperbolic X is divisorially hyperbolic, and therefore the following implication holds:

Theorem 1.1. — *Let X be a compact complex manifold. The following implications hold:*

$$\begin{array}{ccccc}
X \text{ is } \textbf{Kähler hyperbolic} & \Longrightarrow & X \text{ is } \textbf{SKT hyperbolic} & \Longrightarrow & X \text{ is } \textbf{Kobayashi hyperbolic} \\
\Downarrow & & & & \Downarrow \\
X \text{ is } \textbf{balanced hyperbolic} & \Longrightarrow & X \text{ is } \textbf{Gauduchon hyperbolic} & \Longrightarrow & X \text{ is } \textbf{divisorially hyperbolic} \\
\Uparrow & & & & \\
X \text{ is } \textbf{degenerate balanced} & & & &
\end{array}$$

Our second main result is to prove a vanishing theorem for the L^2 harmonic spaces on the universal cover of an **SKT hyperbolic** manifold. With conditions less stringent than Kähler hyperbolicity, we obtain the same result as in 1.4.A. Theorem in [7].

2. Aeppli cohomology and SKT metrics

Given a compact complex n-dimensional manifold X, recall that the Bott–Chern and Aeppli cohomology groups of any bidegree (p,q) of X are classically defined, using the spaces $C^{r,s}(X) = C^{r,s}(X,\mathbb{C})$ of smooth $\mathbb{C}$-valued (r,s)-forms on X, as

$$H^{p,q}_{BC}(X,\mathbb{C}) = \frac{\ker(\partial : C^{p,q}(X) \to C^{p+1,q}(X)) \cap \ker(\bar\partial : C^{p,q}(X) \to C^{p,q+1}(X))}{\mathrm{Im}(\partial\bar\partial : C^{p-1,q-1}(X) \to C^{p,q}(X))}$$

$$H^{p,q}_{A}(X,\mathbb{C}) = \frac{\ker(\partial\bar\partial : C^{p,q}(X) \to C^{p+1,q+1}(X))}{\mathrm{Im}(\partial : C^{p-1,q}(X) \to C^{p,q}(X)) + \mathrm{Im}(\bar\partial : C^{p,q-1}(X) \to C^{p,q}(X))}.$$

The Bott–Chern Laplacian Δ_{BC} and the Aeppli Laplacian Δ_A are the 4^{th} order elliptic differential operators defined, respectively, as

$$\Delta_{BC} := \partial^\star\partial + \bar{\partial}^\star\bar{\partial} + (\partial\bar{\partial})^\star(\partial\bar{\partial}) + (\partial\bar{\partial})(\partial\bar{\partial})^\star + (\partial^\star\bar{\partial})^\star(\partial^\star\bar{\partial}) + (\partial^\star\bar{\partial})(\partial^\star\bar{\partial})^\star,$$

and

$$\Delta_A := \partial\partial^\star + \bar{\partial}\bar{\partial}^\star + (\partial\bar{\partial})^\star(\partial\bar{\partial}) + (\partial\bar{\partial})(\partial\bar{\partial})^\star + (\partial\bar{\partial}^\star)(\partial\bar{\partial}^\star)^\star + (\partial\bar{\partial}^\star)^\star(\partial\bar{\partial}^\star),$$

where $\partial^\star = -\star\bar{\partial}\star$, and $\star = \star_\omega : \Lambda^{p,q}T^\star X \to \Lambda^{n-q,n-p}T^\star X$ is the Hodge-star isomorphism defined by ω for arbitrary $p, q = 0, \ldots, n$. Note that $\star\Delta_{BC} = \Delta_A\star$ and $\Delta_{BC}\star = \star\Delta_A$, then $u \in \ker\Delta_{BC}$ if and only if $\star u \in \ker\Delta_A$.

The Bott–Chern Laplacian is elliptic and essentially self-adjoint, so it induces a three-space decomposition

$$C^\infty_{p,q} = \ker\Delta_{BC} \oplus \mathrm{Im}\partial\bar{\partial} \oplus (\mathrm{Im}\partial^\star + \mathrm{Im}\bar{\partial}^\star)$$

that is orthogonal w.r.t. the L^2 scalar product defined by ω. We have

$$\ker\partial \cap \ker\bar{\partial} = \ker\Delta_{BC} \oplus \mathrm{Im}\partial\bar{\partial}.$$

We also have

$$\mathrm{Im}\Delta_{BC} = \mathrm{Im}\partial\bar{\partial} \oplus (\mathrm{Im}\partial^\star + \mathrm{Im}\bar{\partial}^\star).$$

Similarly, the 4^{th} order Aeppli Laplacian is elliptic and essentially self-adjoint, so it induces a three-space decomposition

$$C^\infty_{p,q} = \ker\Delta_A \oplus \mathrm{Im}(\partial\bar{\partial})^\star \oplus (\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}),$$

that is orthogonal w.r.t. the $L2$ scalar product defined by ω. We have

$$\ker(\partial\bar{\partial}) = \ker\Delta_A \oplus (\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}). \tag{1}$$

We also have

$$\mathrm{Im}\Delta_A = \mathrm{Im}(\partial\bar{\partial})^\star \oplus (\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}).$$

2.1. Cones of classes of metrics. — Recall the classical notion introduced by Popovici [15, Definition 5.1]: the Gauduchon cone $\mathcal{G}_X$ of a compact complex manifold X is the set of all Aeppli cohomology classes of Gauduchon metrics on X. As such, it is an open convex cone in $H^{n-1,n-1}_A(X,\mathbb{R})$, that generalises the Kähler cone $\mathcal{K}_X$. Moreover, it is never empty, thanks to the existence of the Gauduchon metric on all compact complex manifolds.

Recall that a Hermitian metric ω on X is said to be an *SKT metric* if $\partial\bar{\partial}\omega = 0$. Any such metric ω defines an Aeppli cohomology class associated with ω. We start by introducing the following analogue of the Gauduchon cone in bidegree $(1,1)$.

Definition 2.1. — Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The **SKT cone** of X is the set:

$$\mathcal{SKT}_X := \big\{[\omega]_A \in H^{1,1}_A(X,\mathbb{R}) \mid \omega \text{ is a SKT metric on } X\big\} \subset H^{1,1}_A(X,\mathbb{R}).$$

Any element $[\omega]_A$ of the SKT cone $\mathcal{SKT}_X$ is called an **Aeppli-SKT class**.

We will also introduce the *cone in cohomology of bidegree* $(n-1,n-1)$ of X, which generalises the pseudo-effective cone $\mathcal{E}_X$ of Bott–Chern cohomology classes of d-closed semi-positive $(1,1)$-currents introduced by Demailly [4].

Definition 2.2. — Let X be a compact complex manifold. The **co-pseudo-effective cone** of X is the set

$$\mathcal{E}^{n-1,n-1}_X := \big\{[T]_{BC} \in H^{n-1,n-1}_{BC}(X,\mathbb{R}) \mid T \geq 0 \ d\text{-closed} \\ (n-1,n-1)\text{-current on } X\big\}.$$

Any element $[T]_{BC}$ of the co-pseudo-effective cone $\mathcal{E}^{n-1,n-1}_X$ is called a **co-pseudo-effective class**.

Proposition 2.3. — *Let X be a compact complex manifold. Then*

(i) *The SKT cone $\mathcal{SKT}_X$ is an open convex subset of $H^{1,1}_A(X,\mathbb{R})$.*
(ii) *The set $\mathcal{E}^{n-1,n-1}_X$ is a closed convex cone in $H^{n-1,n-1}_{BC}(X,\mathbb{R})$.*
(iii) *The following two statements hold*
  (a) *Given any class $\mathfrak{c}^{n-1,n-1}_{BC} \in H^{n-1,n-1}_{BC}(X,\mathbb{R})$, the following equivalence holds:*

$$\mathfrak{c}^{n-1,n-1}_{BC} \in \mathcal{E}^{n-1,n-1}_X \\ \iff \mathfrak{c}^{n-1,n-1}_{BC}.\mathfrak{c}^{1,1}_A \geq 0 \text{ for every class } \mathfrak{c}^{1,1}_A \in \mathcal{SKT}_X.$$

  (b) *Given any class $\mathfrak{c}^{1,1}_A \in H^{1,1}_A(X,\mathbb{R})$, the following equivalence holds:*

$$\mathfrak{c}^{1,1}_A \in \overline{\mathcal{SKT}}_X \\ \iff \mathfrak{c}^{n-1,n-1}_{BC}.\mathfrak{c}^{1,1}_A \geq 0 \text{ for every class } \mathfrak{c}^{n-1,n-1}_{BC} \in \mathcal{E}^{n-1,n-1}_X.$$

Proof. — (i) If ω_1 and ω_2 are **SKT** metrics on X, so is any linear combination $\lambda\omega_1 + \mu\omega_2$ with λ, μ non-negative reals. Therefore, $\mathcal{SKT}_X$ is a convex cone. To show that $\mathcal{SKT}_X$ is an open subset of $H^{1,1}_A(X,\mathbb{R})$, let us equip the finite-dimensional vector space $H^{1,1}_A(X,\mathbb{R})$ with an arbitrary norm $\|\ \ \|$ (e.g., the Euclidian norm after we have fixed a basis; at any rate, all the norms are equivalent). Let $[\omega]_A \in \mathcal{SKT}_X$ be an arbitrary element, where $\omega > 0$ is some SKT metric on X. Let $\alpha \in H^{1,1}_A(X,\mathbb{R})$ be a class such that $\|\alpha - [\omega]_A\| < \epsilon$ for some small $\epsilon > 0$. Fix any Hermitian metric $\widetilde{\omega}$ on X and consider the Aeppli Laplacian Δ_A defined by $\widetilde{\omega}$ inducing the Hodge isomorphism $H^{1,1}_A(X,\mathbb{R}) \simeq$