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COMPLEX ANALYTIC GEOMETRY OF COMPLEX PARALLELIZABLE MANIFOLDS

Jörg Winkelmann

Abstract. — We investigate complex parallelizable manifolds, *i.e.*, complex manifolds arising as quotients of complex Lie groups by discrete subgroups. Special emphasis is put on quotients by discrete subgroups which are cocompact or at least of finite covolume.

These quotient manifolds are studied from a complex-analytic point of view. Topics considered include submanifolds, vector bundles, cohomology, deformations, maps and functions. Furthermore arithmeticity results for compact complex nilmanifolds are deduced.

An exposition of basic results on lattices in complex Lie groups is also included, in order to improve accessibility.

***Résumé* (Géométrie analytique complexe et variétés complexes parallélisables)**

On étudie les variétés complexes parallélisables, c'est-à-dire les variétés quotients des groupes de Lie complexes par des sous-groupes discrets. On s'intéresse tout particulièrement aux quotients par des sous-groupes discrets cocompacts ou de covolume fini.

Ces variétés quotients sont étudiées du point de vue de la géométrie analytique complexe. On traite notamment les sujets suivants : les sous-variétés, les fibrés vectoriels, la cohomologie, les déformations, les applications et les fonctions. De plus, on en déduit des résultats d'arithméticité pour des nil-variétés complexes compactes.

Pour faciliter la lecture du texte, on a inclus un exposé de résultats de base sur les réseaux dans les groupes de Lie complexes.

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PREFACE

This book is concerned with quotients of complex Lie groups by discrete subgroups. Of particular interest are those quotients where the discrete subgroup is large in a certain sense, e.g. cocompact or of finite covolume. We are interested in the complex-analytic properties of these quotient manifolds. Typical questions include those on the existence of meromorphic, holomorphic or plurisubharmonic functions, existence and structure of complex subspaces and vector bundles, the rigidity of the complex structure and the nature of certain cohomology groups. Arithmeticity results for complex nilmanifolds are derived.

The core material for this book comes from the "Habilitationsschrift" of the author ([**153**]). However, there are a number of changes. We provide a more complete exposition of the basic techniques used in this area and include proofs for auxiliary results wherever possible without too much of an effort. In the Habilitationsschrift mainly only the quotients by *cocompact* discrete subgroups were considered. Although substantial additional work was required, most results of the Habilitationsschrift could be proved for lattices, *i.e.*, discrete subgroups with finite covolume. These have been included in the present work.

I wish to express my gratitude and indebtedness to Alan T. Huckleberry for his consistent support throughout my mathematical development.

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Finally, I wish express my gratitude to the Société Mathématique de France for the opportunity to publish this book in their "Memoirs".

CHAPTER 1

INTRODUCTION

1.1. Summary

This book is organized in the following way. In the first chapter we introduce some basic notions, provide some first examples of complex parallelizable manifolds and collect some general results on Haar measures and on subgroups of linear groups.

In the second chapter we discuss arithmetic groups. We cite arithmeticity and superrigidity results and a criterion for the cocompactness of arithmetic groups. Furthermore, some examples of arithmetic lattices are presented.

In the third chapter a number of foundational results on lattices in complex Lie groups are presented. Particular emphasis is given to density results and results implying that certain orbits are closed. Moreover, the Albanese torus and the algebraic reduction is discussed and it is shown that compact complex tori are the only quotients of complex Lie groups by lattices which admit a Kähler metric.

In the fourth chapter, complex analytic subspaces of quotients of complex Lie groups by lattices are studied. Normally, there are no hypersurfaces, but there always exists closed analytic subspaces of higher codimension if the group is non-commutative. We discuss the Kodaira-dimension of subspaces and show that in a certain way it closely reflects the behaviour of subspaces in tori.

In the fifth chapter, holomorphic mappings and automorphism groups of complex parallelizable manifolds are investigated. Often holomorphic mappings are automatically equivariant. This permits a complete description of self-maps in general and the automorphism group in particular.

Chapter 6 is concerned with basic facts on homogeneous vector bundles and conditions under which a homogeneous vector bundle must be flat.

In Chapter 7 we deduce classification results for flat vector bundles over parallelizable manifolds. In particular we introduce the notion of *essentially antiholomorphic* representations and prove that every flat vector bundle over a quotient of a semisimple complex Lie group without rank-1-factor by a lattice is induced by an essentially antiholomorphic representation. Furthermore, the topological structure of homogeneous vector bundles is investigated and it is deduced that every positive-dimensional compact complex manifold admits a non-trivial holomorphic vector bundle.

In Chapter 8 we study deformations of the complex structure of compact complex parallelizable manifolds and investigate the cohomology group $H^1(\cdot, \mathcal{O})$. Forthermore