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REVISITING DERIVED CRYSTALLINE COHOMOLOGY

by Zhouhang Mao

Abstract. — We prove that the ∞-category of surjections of animated rings is projectively generated, introduce and study the notion of animated PD-pairs—surjections of animated rings with a "derived" PD-structure. This allows us to generalize classical results to nonflat and nonfinitely generated situations.

Using animated PD-pairs, we develop several approaches to derived crystalline cohomology and establish comparison theorems. As an application, we generalize the comparison between derived and classical crystalline cohomology from syntomic (affine) schemes (due to Bhatt) to quasisyntomic schemes.

We also develop a noncompleted animated analogue of prisms and prismatic envelopes. We prove a variant of the Hodge–Tate comparison for animated prismatic envelopes from which we deduce a result about flat cover of the final object for quasisyntomic schemes, which generalizes several known results under smoothness and finiteness conditions.

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Résumé (*Revisiter la cohomologie cristalline dérivée*). — Nous prouvons que la ∞-catégorie des surjections d'anneaux animés est projectivement générée, introduisons et étudions la notion de PD-paires animées – des surjections d'anneaux animés avec une PD-structure "dérivée". Cela nous permet de généraliser des résultats classiques à des situations non plates et non de type fini.

En utilisant les PD-paires animées, nous développons plusieurs approches de la cohomologie cristalline dérivée et établissons des théorèmes de comparaison. En tant qu'application, nous généralisons la comparaison entre la cohomologie cristalline dérivée et classique à partir de schémas syntomiques (affines) (due à Bhatt) à des schémas quasisyntomiques.

Nous développons également un analogue animé non complété des prismes et des enveloppes prismatiques. Nous prouvons une variante de la comparaison de Hodge-Tate pour les enveloppes prismatiques animées, à partir de laquelle nous déduisons un résultat sur le recouvrement plat de l'objet final pour les schémas quasisyntomiques, qui généralise plusieurs résultats connus sous des conditions de lissité et de finitude.

1. Introduction

In this introductory section, we start with a nontechnical discussion of the background, stating the main results in simplified forms. Then we explain the main techniques used in this article. After that, we present the main definitions and constructions used in this article.

1.1. Background and main results. — In this section, we discuss the background and the main results of the current work in a simplified form.

Regular sequences and local complete intersections play an important role in the study of Noetherian rings. However, in arithmetic geometry, Noetherianness is not preserved by operations related to perfectoids. Various generalizations to the non-Noetherian case are available. In [11], it has been shown that, the *quasiregularity* (à la Quillen) is a particularly good candidate to replace (Koszul) regularity in classical algebraic geometry: an ideal I of a ring A is called *quasiregular* (Definition 3.53) if the A/I-module I/I^2 is flat and the homotopy groups $\pi_i(L_{(A/I)/A})$ of the cotangent complex vanish for $i > 1$, or equivalently put, $L_{(A/I)/A} \simeq (I/I^2)[1]$. In particular, if an ideal is generated by a Koszul-regular sequence, then it is also quasiregular.

Let us briefly review some details in the simple case of characteristic p (instead of mixed characteristic). An $\mathbb{F}_p$-algebra R is called *perfect* if the Frobenius map $R \to R, x \mapsto x^p$ is bijective. An $\mathbb{F}_p$-algebra S is called *quasiregular semiperfect* if there exists a perfect $\mathbb{F}_p$-algebra R along with a surjective map $R \twoheadrightarrow S$ of rings of which the kernel $I \subseteq R$ is quasiregular. In this case, [11, Thm 8.12] shows that the derived de Rham cohomology of R with respect to the base $\mathbb{F}_p$ is concentrated in degree 0, and as a ring, it is equivalent to the

PD-envelope of (R, I). Since the cotangent complex $L_{R/\mathbb{F}_p}$ vanishes, the base $\mathbb{F}_p$ of the derived de Rham cohomology can be replaced by R.

This result was already known [9, Thm 3.27] when the kernel I of the map $R \twoheadrightarrow S$ in question is Koszul regular. In other words, [11] generalizes the classical results about Koszul-regular ideals to quasiregular ideals.

In this article, we develop a different approach, which works with greater generality: we do not need the base to be perfect, of characteristic p or even "p-local"Noetherian rings. such as $\mathbb{Z}_p$ or a perfectoid ring. We build a machinery to extend results about Koszul-regular ideals to quasiregular ideals in a systematic fashion. We say that a map $R \to S$ of animated rings [14, §5.1] is *surjective* if the induced map $\pi_0(R) \to \pi_0(S)$ is surjective (Definition 3.27).

THEOREM 1.1 (Theorem 3.29). — *The ∞-category of surjective maps of animated rings is projectively generated. The set $\{\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n] \twoheadrightarrow \mathbb{Z}[x_1, \ldots, x_m] \mid m, n \in \mathbb{N}\}$ of objects forms a set of compact projective generators.*

For technical reasons, we will introduce the ∞-category of *animated pairs*, which is equivalent to the ∞-category of surjective maps of animated rings. By the formalism of left derived functors (Proposition A.14), given a functor defined for "standard" Koszul-regular pairs $(\mathbb{Z}[X, Y], (Y))$ where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$[1], we get a functor defined on *all* animated pairs, and in particular, on classical ring-ideal pairs (A, I), and any comparison map between such functors is determined by the restriction to these Koszul-regular pairs. We learned the importance of such standard pairs from the proof of [8, Cor 4.14].

In order to formulate a reasonable generalization of [11, Thm 8.12], just as we need animated pairs, we also need *animated PD-pairs* (Definition 3.21), denoted by $(A \twoheadrightarrow A'', \gamma)$ (Notation 3.31). There is a canonical forgetful functor from the ∞-category of animated PD-pairs to the ∞-category of animated pairs, which preserves small colimits (Proposition 3.40). This is remarkable since the forgetful functor from the 1-category of PD-pairs to the 1-category of ring-ideal pairs does not preserve small colimits (Remark 3.41). Moreover, the forgetful functor admits a left adjoint, called the *animated PD-envelope functor*.

In general, the animated PD-envelope, considered as a kind of derived functor, is different from the PD-envelope. We will show that, there is a canonical filtration on the animated PD-envelope of $\mathbb{F}_p$-pairs[2] (i.e., pairs (A, I) where A is an $\mathbb{F}_p$-algebra), called the *conjugate filtration* (Definition 3.68), of which we can control the associated graded pieces:

1. In this article, the multivariable notations X and Y are used from time to time.
2. Or more generally, of animated $\mathbb{F}_p$-pairs.

Theorem 1.2 (Corollaries 3.61 and 3.71). — *Let A be an $\mathbb{F}_p$-algebra and $I \subseteq A$ an ideal*[3]*. Then*

1. *The animated PD-envelope of (A, I) admits a natural animated $\varphi_A^*(A/I)$-algebra structure.*
2. *For every $i \in \mathbb{N}$, the $(-i)$-th associated graded piece of the animated PD-envelope of A is, as a $\varphi_A^*(A/I)$-module spectrum, naturally equivalent to $\varphi_A^*(\Gamma^i_{A/I}(L_{(A/I)/A}[-1]))$, where $\Gamma^i_{A/I}$ is the i-th derived divided power.*

As a corollary, the notion of quasiregularity provides an important *acyclicity condition*: along with a mild assumption, the animated PD-envelope coincides with the classical PD-envelope:

Theorem 1.3 (Corollary 3.79). — *Let A be an $\mathbb{F}_p$-algebra, $I \subseteq A$ a quasiregular ideal. Suppose that the (derived) Frobenius twist $(A/I) \otimes^{\mathbb{L}}_{A,\varphi_A} A$ is concentrated in degree 0, i.e., $\operatorname{Tor}^i_A(A/I, A) \cong 0$ (where the last A is viewed as an A-module via the Frobenius $\varphi_A : A \to A$) for all $i \in \mathbb{N}_{>0}$. Then the animated PD-envelope of (A, I) coincides with the classical PD-envelope.*

We want to point out that $(A/I) \otimes^{\mathbb{L}}_{A,\varphi_A} A$ being concentrated in degree 0 is a very mild assumption. For example, when $I \subseteq A$ is generated by a Koszul-regular sequence, then this holds automatically [9, Lem 3.41]. This also happens when (A, I) comes from a "good" PD-envelope;see Remark 4.73. Using this, we show that

Theorem 1.4 (Proposition 3.83). — *Let A be a ring and $I \subseteq A$ an ideal generated by a Koszul-regular sequence. Then the animated PD-envelope of (A, I) coincides with the classical PD-envelope.*

Moreover, this mild assumption is not needed if we are only interested in associated graded pieces of the PD-filtration, which answers a question of Illusie [24, VIII. Ques 2.2.4.2]:

Theorem 1.5 (Propositions 3.90 and 3.98). — *Let A be an $\mathbb{F}_p$-algebra, $I \subseteq A$ a quasiregular ideal. Then there is a canonical comparison map from the animated PD-envelope to the classical PD-envelope (B, J, γ) of (A, I) compatible with PD-filtrations that induces equivalences on associated graded pieces. Furthermore, these associated graded pieces $\operatorname{gr}^*_{\mathrm{PD}} B$, as a graded commutative ring, are given by the free divided power A/I-algebra $\Gamma^*_{A/I}(I/I^2)$ generated by the A/I-module I/I^2.*

The key point is that animated PD-envelopes admit natural PD-filtrations of which we can control the associated graded pieces (Proposition 3.90).

3. In the introduction, for sake of simplicity, we usually replace the occurrences of animated pairs (resp. animated PD-pairs) by ring-ideal pairs (resp. PD-pairs) as input data.

Based on animated PD-pairs, we develop a theory of *derived crystalline cohomology* (Definition 4.21) based on a technical construction called *derived de Rham cohomology of a map of animated PD-pairs* (Definition 4.10) that generalizes the derived de Rham cohomology of a map of animated rings. In other words, our derived crystalline cohomology should be understood as a variant of derived de Rham cohomology, not site-theoretic cohomology. These functors preserve small colimits by Proposition 4.23 and Lemma 4.14; therefore formal properties such as base change compatibility and "Künneth" formula hold (Corollaries 4.24, 4.26 and 4.27).

In fact, the animated PD-envelope is, roughly speaking, a special case of derived crystalline cohomology:

THEOREM 1.6 (Proposition 4.75). — *Let (A, I, γ_A) be a PD-pair and $J \subseteq A$ be an ideal containing I. Let $(B \twoheadrightarrow A/J, \gamma_B)$ be the relative animated PD-envelope of (A, J) with respect to the PD-pair (A, I, γ_A). Then the underlying $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra of B is equivalent to the derived crystalline cohomology of A/J with respect to (A, I, γ_A).*

From this, we deduce a generalization of [11, Thm 8.12] under quasiregularity and the Tor-independent assumption mentioned above. To see this, similarly to the animated PD-envelope, we introduce the *conjugate filtration* on the derived crystalline cohomology (Definition 4.49) and on the relative animated PD-envelope (Definition 4.69) in characteristic p, and we have a similar control of associated graded pieces for the conjugate filtration on relative animated PD-envelopes (Corollary 4.70) and also on the derived crystalline cohomology, which is a crystalline variant of the *Cartier isomorphism* (cf. [9, Prop 3.5] over the base $(A, I, \gamma) = (\mathbb{F}_p, 0, 0)$):

THEOREM 1.7 (Proposition 4.54). — *Let (A, I, γ) be a PD-pair, where A is an $\mathbb{F}_p$-algebra. Note that the Frobenius map $\varphi_A : A \to A$ factors through $A \twoheadrightarrow A/I$, giving rise to a natural map $\varphi_{(A,I)} : A/I \to A$ (cf. Lemma 4.42). Then for every animated A/I-algebra R and $n \in \mathbb{N}$, the $(-i)$-th associated graded piece of the conjugate filtration on the derived crystalline cohomology of R relative to (A, I, γ) is, as a $\varphi^*_{(A,I)}(R)$-module spectrum, equivalent to $\varphi^*_{(A,I)}\left(\bigwedge^i_R L_{R/(A/I)}\right)[-i]$.*

On the other hand, similarly to [5], we develop an *affine crystalline site* (Definition 4.76) based on animated PD-pairs (Bhatt already indicated such a possibility;see the paragraph before [9, Ex 3.21]). Recall that a map $A \to R$ of rings is called *quasisyntomic* (Definition 4.96) if it is flat and the cotangent complex $L_{R/A}$, as an R-module spectrum, has Tor-amplitude in $[0, 1]$. We could also compare the derived crystalline cohomology to the site-theoretic cohomology: