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AUTOUR DES MOTIFS

École d'été franco-asiatique de géométrie algébrique et de théorie des nombres

Asian-French summer school on algebraic geometry and number theory

Volume III

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J.-B. Bost et J.-M. Fontaine, éditeurs

***Abstract.* —** This volume contains the third part of the lectures notes of the *Asian-French summer school on algebraic geometry and number theory*, which was held at the Institut des Hautes Études Scientifiques (Bures-sur-Yvette) and the université Paris-Sud XI (Orsay) in July 2006. This summer school was devoted to the theory of motives and its recent developments, and to related topics, notably Shimura varieties and automorphic representations.

***Résumé.* —** Ce volume contient la troisième partie des notes de cours de l'*École d'été franco-asiatique de géométrie algébrique et de théorie des nombres*, qui s'est tenue à l'Institut des Hautes Études Scientifiques (Bures-sur-Yvette) et à l'université Paris-Sud XI en juillet 2006. Cette école était consacrée à la théorie des motifs et à ses récents développements, ainsi qu'à des sujets voisins, comme la théorie des variétés de Shimura et des représentations automorphes.

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Motifs et représentations automorphes

Ce texte est un exposé des relations connues et conjecturées entre motifs des variétés algébriques sur les corps de nombres (au sens de Grothendieck) et représentations ou formes automorphes.

Motifs purs, motifs mixtes et extensions de motifs associés aux surfaces singulières

On rappelle d'abord la construction du motif de Chow sous-jacent à la cohomologie d'intersection d'une surface propre $\overline{X}$, et l'on en étudie les propriétés fondamentales. En utilisant le langage des motifs effectifs géométriques à la Voevodsky, on étudie ensuite le motif du diviseur exceptionnel D dans un éclatement non-singulier de $\overline{X}$. Si toutes les composantes géométriques de D sont de genre zéro, alors le formalisme de Voevodsky permet la construction de certaines extensions de motifs, comme sous-quotients canoniques du motif à support compact de la partie lisse de $\overline{X}$. Dans le cas des surfaces de Hilbert-Blumenthal, ceci donne une interprétation motivique d'une construction récente due à A. Caspar.

Dans cet article, on étudie les propriétés structurales principales de la catégorie triangulée des motifs d'Artin-Tate sur un corps parfait k. On analyse d'abord sa structure de poids, utilisant les résultats principaux de [Bondarko, 2010]. Puis, on étudie sa t-structure, quand k est algébrique sur $\mathbb{Q}$; ceci généralise les résultats principaux de [Levine, 1993]. Enfin, on précise l'interaction de la structure de poids et de la t-structure. Quand k est un corps de nombres, ceci donne un critère utile permettant de caractériser la structure de poids à l'aide des réalisations.

ABSTRACTS

An introduction to Galois representations and modular forms

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These notes are based on a series of lectures given at the summer school held on July 17-29, 2006 at IHÉS. The purpose of the lectures is to explain the basic ideas in the geometric construction of the Galois representations associated to elliptic modular forms of weight at least 2.

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We first recall the construction of the Chow motive modelling intersection cohomology of a proper surface $\overline{X}$, and study its fundamental properties. Using Voevodsky's category of effective geometrical motives, we then study the motive of the exceptional divisor D in a non-singular blow-up of $\overline{X}$. If all geometric irreducible components of D are of genus zero, then Voevodsky's formalism allows us to construct certain one-extensions of motives, as canonical sub-quotients of the motive with compact support of the smooth part of $\overline{X}$. Specializing to Hilbert-Blumenthal surfaces, we recover a motivic interpretation of a recent construction of A. Caspar.

In this paper, we study the main structural properties of the triangulated category of Artin-Tate motives over a perfect base field k. We first analyze its weight structure, building on the main results of [Bondarko, 2010]. We then study its t-structure, when k is algebraic over $\mathbb{Q}$, generalizing the main result of [Levine, 1993]. We finally exhibit the interaction of the weight structure and the t-structure. When k is a number field, this will give a useful criterion identifying the weight structure *via* realizations.

FOREWORD

This third volume contains notes of the lectures by T. Saito and L. Clozel and expanded versions of some seminar talks by J. Wildeshaus.

As these contributions have been completed in 2011, we want to indicate a few additional references concerning the topics of these lectures and seminars that might be useful to the reader.

An English version of T. Saito's book quoted as reference [18] in his notes has been published as the two volumes [**7**] and [**8**].

References [8], [14], [20], [35], [39], [54] in the notes of L. Clozel have now been published as [**1**], [**2**], [**4**], [**5**], [**6**], [**13**].

Concerning the construction of systems of Galois representations associated to suitable cuspidal representations for totally real or CM fields discussed in L. Clozel's Lecture 3, besides [49] (now published as [**11**]), some further progresses appear in [**3**], [**10**], and more recently in [**12**] and in [**9**].

These last two contributions constitute spectacular breakthroughs. In [**12**], Harris, Lan, Taylor and Thorne construct Galois representations associated to possibly not self-dual representations. In [**9**], Scholze constructs Galois representations associated to torsion classes in the cohomology of arithmetic manifolds, by investigating the "perfectoid geometry" of Shimura varieties.

Further developments of the constructions of J. Wildeshaus involving boundary and interior motives have appeared in [**14**] and [**15**].

References

[1] T. Barnet-Lamb, T. Gee, D. Geraghty & R. Taylor – "Potential automorphy and change of weight", *Ann. of Math.* **179** (2014), p. 501–609.

[2] A. Caraiani – "Local-global compatibility and the action of monodromy on nearby cycles", *Duke Math. J.* **161** (2012), p. 2311–2413.

[3] G. Chenevier & M. Harris – "Construction of automorphic Galois representations, II", *Camb. J. Math.* **1** (2013), p. 53–73.

[4] L. Clozel – "Purity reigns supreme", *Int. Math. Res. Not.* **2013** (2013), p. 328–346.

[5] V. Lafforgue – "Estimées pour les valuations p-adiques des valeurs propes des opérateurs de Hecke", *Bull. Soc. Math. France* **139** (2011), p. 455–477.

[6] G. Laumon & B. C. Ngô – "Le lemme fondamental pour les groupes unitaires", *Ann. of Math.* **168** (2008), p. 477–573.

[7] T. Saito – *Fermat's last theorem*, Translations of Mathematical Monographs, vol. 243, Amer. Math. Soc., Providence, RI, 2013.

[8] ______, *Fermat's last theorem*, Translations of Mathematical Monographs, vol. 245, Amer. Math. Soc., Providence, RI, 2014.

[9] P. Scholze – "On torsion in the cohomology of locally symmetric varieties", *Ann. of Math.* **182** (2015), p. 945–1066.

[10] P. Scholze & S. W. Shin – "On the cohomology of compact unitary group Shimura varieties at ramified split places", *J. Amer. Math. Soc.* **26** (2013), p. 261–294.

[11] S. W. Shin – "Galois representations arising from some compact Shimura varieties", *Ann. of Math.* **173** (2011), p. 1645–1741.

[12] M. Taylor, K.-W. Harris, R. Lan & J. Thorne – "On the rigid cohomology of certain Shimura varieties", preprint arXiv:1411.6717.

[13] R. Taylor & T. Yoshida – "Compatibility of local and global Langlands correspondences", *J. Amer. Math. Soc.* **20** (2007), p. 467–493.

[14] J. Wildeshaus – "On the interior motive of certain Shimura varieties: the case of Hilbert-Blumenthal varieties", *Int. Math. Res. Not.* **2012** (2012), p. 2321–2355.

[15] ______, "On the interior motive of certain Shimura varieties: the case of Picard surfaces", *Manuscripta Math.* **148** (2015), p. 351–377.

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AN INTRODUCTION TO GALOIS REPRESENTATIONS AND MODULAR FORMS

by

Takeshi Saito

Abstract. – These notes are based on a series of lectures given at the summer school held on July 17-29, 2006 at IHÉS. The purpose of the lectures is to explain the basic ideas in the geometric construction of the Galois representations associated to elliptic modular forms of weight at least 2.

Résumé **(Une introduction aux représentations de Galois et aux formes modulaires)**
Ces notes sont basées sur une série de cours donnés à l'École d'été qui a eu lieu du 17 au 29 juillet 2006 à l'IHÉS. Le but de ces cours est d'expliquer les idées fondamentales de la construction géométrique des représentations de Galois associées aux formes modulaires elliptiques de poids supérieur ou égal à 2.

Motivation

Galois representations associated to modular forms play a central role in modern number theory. In this introduction, we give a reason why they take such a position.

A goal in number theory is to understand finite extensions of $\mathbb{Q}$. By Galois theory, it is equivalent to understanding the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One may say that one knows a group if one knows its representations.

Representations are classified by their degrees. Class field theory provides us with a precise understanding of the representations of degree 1, or characters. By the theorem of Kronecker-Weber, a continuous character $G_{\mathbb{Q}} \to \mathbb{C}^\times$ is a Dirichlet character

$$G_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \to (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

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for some integer $N \geq 1$. If we consider not only complex continuous characters but also ℓ-adic characters $G_{\mathbb{Q}} \to \mathbb{Q}_\ell^\times$ for a prime ℓ, we find more characters. For example, the ℓ-adic cyclotomic character is defined as the composition:

$$G_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^n}, n \in \mathbb{N})/\mathbb{Q}) = \varprojlim_n \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^n})/\mathbb{Q}) \to \varprojlim_n (\mathbb{Z}/\ell^n\mathbb{Z})^\times = \mathbb{Z}_\ell^\times \subset \mathbb{Q}_\ell^\times.$$

The ℓ-adic characters "with motivic origin" are generated by Dirichlet characters and ℓ-adic cyclotomic characters:

$$\{\text{“geometric” } \ell\text{-adic character of } G_{\mathbb{Q}}\} \\ = \langle \text{Dirichlet characters, } \ell\text{-adic cyclotomic characters} \rangle$$

if we use a fancy terminology "geometric," that will not be explained in this note. For the definition, we refer to [**11**].

When we leave the realm of class field theory, the first representations we encounter are those of degree 2. For ℓ-adic Galois representations of degree 2, we expect to have (cf. [**11**]) a similar equality

$$\{\text{odd “geometric” } \ell\text{-adic representations of } G_{\mathbb{Q}} \text{ of degree 2} \\ \text{with distinct Hodge-Tate weights}\} = \{\ell\text{-adic representations} \\ \text{associated to modular form of weight at least 2}\},$$

up to a twist by a power of the cyclotomic character. In other words, the Galois representations associated to modular forms are the first ones we encounter when we explore outside the domain of class field theory.

In these notes, we discuss only one direction $\supset$ established by Shimura and Deligne ([**20**], [**4**]). We will not discuss the other direction $\subset$, which is almost established ([**13**]) after the revolutionary work of Wiles, although it has significant consequences including Fermat's last theorem, the modularity of elliptic curves, etc. ([**24**], [**2**]).

In Section 1, we recall the definition of modular forms and state the existence of Galois representations associated to normalized eigen cusp forms. We introduce modular curves defined over $\mathbb{C}$ and over $\mathbb{Z}[\frac{1}{N}]$ as the key ingredient in the construction of the Galois representations, in Section 2. Then, we construct the Galois representations in the case of weight 2 by decomposing the Tate module of the Jacobian of a modular curve in Section 3. In the final Section 4, we briefly sketch an outline of the construction in the higher weight case.

Proofs will be only sketched or omitted mostly. The author apologizes that he also omits the historical accounts completely. Some more detail can be found in the books [**15, 16**].

The author would like to thank the participants of the summer school for pointing out numerous mistakes and inaccuracies during the lectures.

I would like to acknowledge on this occasion that many of Japanese participants of the summer school are supported by JSPS Core-to-Core Program 18005 New Developments of Arithmetic Geometry, Motive, Galois Theory, and Their Practical Applications.

1. Galois representations and modular forms

1.1. Modular forms. – Let $N \geq 1$ and $k \geq 2$ be integers and $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a character. We will define the $\mathbb{C}$-vector spaces $S_k(N,\varepsilon) \subset M_k(N,\varepsilon)$ of cusp forms and of modular forms of level N, weight k and character ε. We will see later in §2.4 that they are of finite dimension by using compactifications of modular curves. For $\varepsilon = 1$, we write $S_k(N) \subset M_k(N)$ for $S_k(N,1) \subset M_k(N,1)$. For this subsection, we refer to **[10]** Chapter 1.

A subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is called a congruence subgroup if there exists an integer $N \geq 1$ such that $\Gamma \supset \Gamma(N) = \mathrm{Ker}(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}))$. In this note, we mainly consider the congruence subgroups

$$\begin{aligned} &\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a \equiv 1, c \equiv 0 \bmod N \right\} \\ \subset\quad &\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}. \end{aligned}$$

We identify the quotient $\Gamma_0(N)/\Gamma_1(N)$ with $(\mathbb{Z}/N\mathbb{Z})^\times$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d \bmod N$. Their indices are given by

$$\begin{aligned} [SL_2(\mathbb{Z}) : \Gamma_0(N)] &= \prod_{p|N} (p+1)p^{\mathrm{ord}_p(N)-1} = N \prod_{p|N} \left(1 + \frac{1}{p}\right), \\ [SL_2(\mathbb{Z}) : \Gamma_1(N)] &= \prod_{p|N} (p^2-1)p^{2(\mathrm{ord}_p(N)-1)} = N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right). \end{aligned}$$

The action of $SL_2(\mathbb{Z})$ on the Poincaré upper half plane $H = \{\tau \in \mathbb{C} | \mathrm{Im}\ \tau > 0\}$ is defined by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\tau \in H$. For a holomorphic function f on H, we define a holomorphic function $\gamma_k^* f$ on H by

$$\gamma_k^* f(\tau) = \frac{1}{(c\tau + d)^k} f(\gamma\tau).$$

If $k = 2$, we have $\gamma^*(f d\tau) = \gamma_2^*(f) d\tau$.

Definition 1.1. – *Let $\Gamma \supset \Gamma(N)$ be a congruence subgroup and $k \geq 2$ be an integer. We say that a holomorphic function $f : H \to \mathbb{C}$ is a modular form (resp. a cusp form) of weight k with respect to Γ, if the following conditions* (1) *and* (2) *are satisfied.*

(1) *$\gamma_k^* f = f$ for all $\gamma \in \Gamma$.*

(2) *For each $\gamma \in SL_2(\mathbb{Z})$, $\gamma_k^* f$ satisfies $\gamma_k^* f(\tau + N) = \gamma_k^* f(\tau)$ and hence we have a Fourier expansion $\gamma_k^* f(\tau) = \sum_{n=-\infty}^{\infty} a_{\frac{n}{N}}(\gamma_k^* f) q_N^n$ where $q_N = \exp(2\pi i \frac{\tau}{N})$. We require the condition*

$$a_{\frac{n}{N}}(\gamma_k^* f) = 0$$