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METRICS WITH CONE SINGULARITIES ALONG NORMAL CROSSING DIVISORS AND HOLOMORPHIC TENSOR FIELDS

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ABSTRACT. – We prove the existence of non-positively curved Kähler-Einstein metrics with cone singularities along a given simple normal crossing divisor of a compact Kähler manifold, under a technical condition on the cone angles, and we also discuss the case of positively-curved Kähler-Einstein metrics with cone singularities. As an application we extend to this setting classical results of Lichnerowicz and Kobayashi on the parallelism and vanishing of appropriate holomorphic tensor fields.

RÉSUMÉ. – Dans cet article, nous prouvons l'existence de métriques de Kähler-Einstein à courbure négative ayant des singularités coniques le long d'un diviseur à croisements normaux simples sur une variété kählérienne compacte, sous une hypothèse technique sur les angles des cones. Nous discutons également du cas des métriques de Kähler-Einstein à courbure strictement positive avec des singularités coniques. Nous en déduisons que les résultats classiques de Lichnerowicz et Kobayashi sur le parallélisme et l'annulation des champs de tenseurs holomorphes s'étendent à notre cadre.

1. Introduction

Let X be a n-dimensional compact Kähler manifold, and let $D = \sum a_i D_i$ be an effective $\mathbb{R}$-divisor with strictly normal crossing support, such that for all i, $0 < a_i < 1$. In the terminology of the Minimal Model Program, the pair (X, D) is called a *log-smooth klt pair*; following [10], [11], we may also call it a *smooth geometric orbifold*.

One may define for such a pair the notion of *cone metric*, or also *metric with cone singularities along D*: it corresponds to an equivalence class (up to quasi-isometry) of (Kähler) metrics ω_{cone} on $X_0 = X \setminus \operatorname{Supp}(D)$ having the following property: there exists $C > 0$ such that for every point p where $\operatorname{Supp}(D) \cap \Omega = (z^1 \cdots z^d = 0)$, for Ω a coordinate chart near p, we have:

$$C^{-1}\omega_{\text{o}} \leqslant \omega_{\text{cone}} \leqslant C\,\omega_{\text{o}}$$

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where

$$\omega_o = \sqrt{-1}\left(\sum_{i=1}^{d} \frac{dz^i \wedge d\bar{z}^i}{|z^i|^{2a_i}} + \sum_{i=d+1}^{n} dz^i \wedge d\bar{z}^i\right)$$

is the standard cone metric on $\mathbb{C}^n \setminus \operatorname{Supp}(D)$ with respect to the divisor

$$D = \sum_{i=1}^{d} a_i[z^i = 0].$$

In the context of geometric orbifolds the notion of *holomorphic tensors* was first formulated in [10], [11]; as we will see here, it corresponds to holomorphic tensors on $X_0 = X \setminus \operatorname{Supp}(D)$ (in the usual sense) which are bounded with respect to a cone metric along D. Roughly speaking, the results of this paper consist in constructing cone metrics with prescribed Ricci curvature by means of Monge-Ampère equations, and use them to show the vanishing or parallelism of some holomorphic tensors on (X, D) according to the sign of the adjoint bundle $K_X + D$.

We offer next a more extensive presentation of the main theorems obtained in this article. In the following, we fix a log-smooth klt pair (X, D) where we choose to write D in the form

$$D := \sum_{j \in J} (1 - \tau_j) Y_j$$

for some smooth hypersurfaces Y_j having strictly normal intersections, and some real numbers $0 < \tau_j < 1$. The numbers τ_j (or more precisely $2\pi\tau_j$) have a geometric interpretation in terms of the cone angles. As we said, our goal is to construct a Kähler metric ω_∞ on $X_0 := X \setminus (\bigcup Y_j)$ whose Ricci curvature is given according to the sign of the $\mathbb{R}$-divisor $K_X + D$, with prescribed asymptotic along D. For example, let us consider the case where $K_X + D$ is ample. Then we want ω_∞ to satisfy the two following properties:

- ω_∞ is Kähler-Einstein on X_0: $-\operatorname{Ric}(\omega_\infty) = \omega_\infty$ on X_0
- ω_∞ has cone singularities along D.

This problem, in a more general form not involving positivity on the adjoint bundle $K_X + D$, can be restated in the Monge-Ampère setting. Namely, given a Kähler form ω on X, we want to solve the following equation (in φ):

$$\text{(MA)} \qquad (\omega + dd^c\varphi)^n = e^{f + \lambda\varphi} \mu_D$$

for $\lambda \in \{0, 1\}$, $f \in \mathscr{C}^\infty(X)$, and where μ_D is the volume form on X given by

$$\mu_D = \frac{\omega^n}{\prod_{j \in J} |s_j|^{2(1-\tau_j)}}$$

for sections s_j of (L_j, h_j) defining Y_j; moreover, if $\lambda = 0$, one assumes that $\int_X e^f \mu_D = \int_X \omega^n$. Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$.

When $\lambda = 1$, using an elementary regularization argument, one can construct a (unique) continuous solution φ of equation (MA), cf. Section 5.1. The case where $\lambda = 0$ is deeper: as μ_D as L^p density for some $p > 1$ (this is an important place where the klt condition is used), a theorem of S. Kołodziej [24] shows that this equation has a unique (normalized) solution φ_∞, which is continuous on X. By [16, Theorem B] (see also [29]), φ_∞ is known to be smooth outside $\operatorname{Supp}(D)$.

The metric $\omega_\infty = \omega + dd^c\varphi_\infty$ on $X_0 = X \setminus (\bigcup Y_j)$ satisfies the condition stated in the first item. Unfortunately, the theorem of S. Kołodziej or their generalizations do not give us order 2 information on φ_∞ near $\mathrm{Supp}(D)$, which is exactly what the condition in the second item ("cone singularities") requires.

Our main result is the following:

Theorem A (Main Theorem). – *We assume that the coefficients of D satisfy the inequalities*

$$0 < \tau_j \leqslant \frac{1}{2}.$$

Then the Kähler metric ω_∞ solution to the following equation

$$(\omega + dd^c\varphi)^n = \frac{e^{f+\lambda\varphi}}{\prod_{j\in J} |s_j|^{2(1-\tau_j)}}\,\omega^n \tag{MA}$$

has cone singularities along D.

This problem has already been studied in many important particular cases. Indeed, in the standard orbifold case, corresponding to coefficients $\tau_j = \frac{1}{m_j}$ for some integers $m_j \geqslant 2$, G. Tian and S.-T. Yau have established in [32] the existence of such a Kähler-Einstein metric compatible with the (standard) orbifold structure, in the case where $K_X + D$ is ample. R. Mazzeo ([28]) announced the existence of Kähler-Einstein metrics with cone singularities when D is smooth and irreducible (assuming $K_X + D$ ample) while T. Jeffres ([19]) studied the uniqueness under the same assumptions. Moreover the recent article by S. Donaldson (cf. [15] and the references therein) is very much connected with the result above. We refer to the even more recent papers by S. Brendle [9] and R. Berman [3] for the complete analysis of the Ricci-flat (resp. positive Ricci curvature) case under the assumption that D had only one smooth component. After the first version of this article appeared, T. Jeffres, R. Mazzeo and Y. Rubinstein gave in [20] a complete treatment of the Kähler-Einstein problem for metrics with cone singularities along one smooth divisor.

We note that the assumption $\tau_j \leqslant \frac{1}{2}$ is automatically satisfied in the orbifold case, and that it also appears in a crucial way in [9] so as to bound the holomorphic bisectional curvature of the cone metric outside the aforesaid hypersurface.

We discuss now briefly our approach to the proof of this result. The strategy is to regularize the equation (MA) and to obtain uniform estimates; then ω_∞ will be obtained as a limit point of solutions of the regularized equations. In order to achieve this goal, we will proceed as follows. We first approximate the standard cone metric ω_o (or better say, its global version) with a sequence of smooth Kähler metrics $(\omega_\varepsilon)_{\varepsilon>0}$ on X.

The approximations are constructed such that $\omega_\varepsilon := \omega + dd^c\psi_\varepsilon$ belongs to a fixed cohomology class $[\omega]$ for some metric ω on X. The explicit expression of ψ_ε is given in Section 3. Our candidate for the sequence converging to the metric ω_∞ we seek will be

$$\omega_{\varphi_\varepsilon} := \omega_\varepsilon + dd^c\varphi_\varepsilon$$

where φ_ε is the solution to the following Monge-Ampère equation, which may be seen as a regularization of equation (MA):

$$(\star_\varepsilon) \qquad \omega_{\varphi_\varepsilon}^n = \frac{e^{f+\lambda(\psi_\varepsilon+\varphi_\varepsilon)}}{\prod_{j=1}^d (\varepsilon^2+|s_j|^2)^{1-\tau_j}}\omega^n.$$

Here we have $\lambda \in \{0,1\}$, $f \in \mathscr{C}^\infty(X)$ (this function will be given by the geometric context in the second part of this article), and the s_j's are sections of hermitian line bundles (L_j, h_j) such that $Y_j = (s_j = 0)$. If $\lambda = 0$, we impose moreover the normalization

$$\int_X \varphi_\varepsilon \, dV_\omega = 0.$$

First we remark that if $\varepsilon > 0$, then we can solve the equation $(\star_\varepsilon)$ and obtain a solution $\varphi_\varepsilon \in \mathscr{C}^\infty(X)$ thanks to the fundamental theorem of Yau in [35]. Indeed, this can be done since ω_ε is a genuine metric, i.e., it is smooth. Of course, the main part of our work is to analyze the uniformity properties of the family of functions

$$(39) \qquad (\varphi_\varepsilon)_{\varepsilon>0}$$

as $\varepsilon \to 0$. To this end, we mimic the steps of the “closedness” part of the method of continuity in [35], as follows.

– Using the results of [24], we already obtain $\mathscr{C}^0$ estimates; this combined with standard results in the theory of Monge-Ampère equations gives us *interior* $\mathscr{C}^{2,\alpha}$ estimates *provided that* global $\mathscr{C}^2$ estimates have been already established. If we fulfill this program, then we can extract from $(\omega_{\varphi_\varepsilon})_{\varepsilon>0}$ a subsequence converging to the desired solution ω_∞, which will henceforth be smooth outside the support of D.
– As we mentioned earlier, we aim to compare ω_∞ and ω_o, and to this end we need *global* $\mathscr{C}^2$ estimates on φ_ε, i.e., to compare $\omega_{\varphi_\varepsilon}$ and ω_ε in a uniform manner. The key observation, though rather simple, is that in our situation, we only need to obtain a uniform lower bound on the holomorphic curvature of ω_ε so as to get the estimates. In the next section, we detail the preceding observation, providing a general context under which one may obtain such $\mathscr{C}^2$ estimates.

In conclusion, we do not have to deal directly with the singular metric ω_o since we are using its regularization family $(\omega_\varepsilon)_{\varepsilon>0}$: it is thanks to this simple approach that we can avoid the “openness” part of the continuity method, and it enables us to treat the case where D is not necessarily irreducible; compare with [15], [9], [3] and the references therein (especially [21], [2]).

We study then Equation (MA) in the case where $\lambda = -1$. We know that this equation (even when the volume form is smooth) does not necessarily admit a solution, so that we cannot use the same techniques as previously. Therefore, we will only consider the cases where we already know that (MA) admits a solution. More generally, we prove the following result:

THEOREM B. – *Let X be a compact Kähler manifold and $D = \sum(1-\tau_j)Y_j$ a divisor with simple normal crossings such that its coefficients satisfy $0 < \tau_j \leqslant 1/2$. Let $\mu_D = dV/\prod_j |s_j|^{2(1-\tau_j)}$ be a volume form with cone singularities along D, ψ a bounded quasi-psh function, and ω a Kähler form on X. Then any (bounded) solution φ of*

$$(\omega + dd^c\varphi)^n = e^{-\psi}\mu_D$$