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CONTROL AND MIXING FOR 2D NAVIER-STOKES EQUATIONS WITH SPACE-TIME LOCALISED NOISE

BY ARMEN SHIRIKYAN

ABSTRACT. – We consider randomly forced 2D Navier-Stokes equations in a bounded domain with smooth boundary. It is assumed that the random perturbation is non-degenerate, and its law is periodic in time and has a support localised with respect to space and time. Concerning the unperturbed problem, we assume that it is approximately controllable in infinite time by an external force whose support is included in that of the random force. Under these hypotheses, we prove that the Markov process generated by the restriction of solutions to the instants of time proportional to the period possesses a unique stationary distribution, which is exponentially mixing. The proof is based on a coupling argument, a local controllability property of the Navier-Stokes system, an estimate for the total variation distance between a measure and its image under a smooth mapping, and some classical results from the theory of optimal transport.

RÉSUMÉ. – Nous considérons une perturbation aléatoire du système de Navier-Stokes 2D dans un domaine borné à bord régulier. On suppose que la force aléatoire est non dégénérée et que sa loi est périodique en temps et a un support localisé en espace et en temps. En ce qui concerne le problème non perturbé, on suppose qu'il est approximativement contrôlable en temps infini par une force extérieure dont le support est inclus dans celui de la force aléatoire. Sous ces hypothèses, on montre que le processus de Markov engendré par la restriction des solutions aux instants de temps proportionnels à la période possède une unique distribution stationnaire, qui est exponentiellement mélangeante. La démonstration est basée sur un argument de couplage, une propriété de contrôlabilité locale pour le système de Navier-Stokes, une estimation pour la distance en variation totale entre une mesure et son image par une application lisse et quelques résultats classiques de la théorie du transport optimal.

1. Introduction

The main results of this paper can be summarised as follows: first, suitable controllability properties of a non-linear PDE imply the uniqueness and exponential mixing for the associated stochastic dynamics and, second, these properties are satisfied for 2D Navier-Stokes equations with space-time localised noise. To be precise, let us consider from the very beginning the 2D Navier-Stokes system in a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D:

$$\dot{u} + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \quad x \in D, \tag{1.1}$$

$$u\big|_{\partial D} = 0. \tag{1.2}$$

Here $u = (u_1, u_2)$ and p are unknown velocity and pressure of the fluid, $\nu > 0$ is the viscosity, and f is an external force. Let us assume that f is represented as the sum of two functions h and η, the first of which is a given function that is H^1 smooth in space and time and has a locally bounded norm, while the second is either a control or a random force:

$$f(t,x) = h(t,x) + \eta(t,x). \tag{1.3}$$

In both cases, we assume that η is sufficiently smooth and bounded, and its restriction to any cylinder of the form $J_k \times D$ with $J_k = [k-1, k]$ is localised in both space and time (see below for a more precise description of this hypothesis). Let us denote by $\boldsymbol{n}$ the outward unit normal to the boundary ∂D and introduce the space

$$H = \{u \in L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, \langle u, \boldsymbol{n} \rangle = 0 \text{ on } \partial D\}, \tag{1.4}$$

which will be endowed with the usual L^2 norm $\|\cdot\|$. It is well known that for any $u_0 \in H$ problem (1.1), (1.2) supplemented with the initial condition

$$u(0,x) = u_0(x) \tag{1.5}$$

has a unique solution $u = u(t; u_0, f)$, which is a continuous function of time valued in H.

Our main result concerns the property of exponential mixing for the discrete-time Markov process in H associated with (1.1)–(1.3), and we now present a simplified version of the hypotheses under which it is valid. We assume that the deterministic force h is a 1-periodic function of time whose restriction to any bounded subset of $\mathbb{R} \times D$ is H^1-regular. As for the random force η, we assume that it satisfies the four conditions below. Let $Q \subset J_1 \times D$ be an open set and let $Q_k = \{(t,x) : (t-k+1, x) \in Q\}$.

Localisation: For any integer $k \geq 1$, the restriction of η to the cylinder $J_k \times D$ is supported by Q_k.

Let us denote by $\eta_k(t,x)$ the restriction of $\eta(t+k-1, x)$ to the domain $J_1 \times D$.

Independence: The functions η_k form a sequence of i.i.d. random variables in $H^1(J_1 \times D, \mathbb{R}^2)$ with a law λ.

Non-degeneracy: The measure λ is decomposable in the following sense: there is an orthonormal basis $\{e_j\}$ in the space $L^2(Q, \mathbb{R}^2)$ such that $e_j \in H_0^1(Q, \mathbb{R}^2)$ for all $j \geq 1$, and

$$\eta_k(t,x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(t,x),$$

where ξ_{jk} are independent random variables valued in $[-1,1]$ and $\{b_j\}$ are positive numbers such that $\sum_j b_j \|e_j\|_{H^1} < \infty$. Moreover, the laws of ξ_{jk} possess C^1-smooth densities with respect to the Lebesgue measure on $\mathbb{R}$.

Approximate controllability: There is $\hat{u} \in H$ such that problem (1.1)–(1.3) is approximately controllable to $\hat{u}$ with a control function $\tilde{\eta}$ such that, for any $k \geq 1$, the restriction of $\tilde{\eta}(t+k-1,x)$ to $J_1 \times D$ belongs to $\operatorname{supp}\lambda$, and the time of control can be chosen the same for the initial functions u_0 from a given bounded subset of H.

The 1-periodicity of h and the second of the above hypotheses imply that the restrictions of solutions for (1.1)–(1.3) to integer times form a family of Markov chains in H. The following theorem, which is the main result of the paper, describes the long-time asymptotics of this chain.

MAIN THEOREM. – *Under the above hypotheses, the Markov chain associated with problem* (1.1)–(1.3) *has a unique stationary measure μ. Moreover, there are positive constants C and γ such that, for any* 1*-Lipschitz function $F : H \to \mathbb{R}$ and any $u_0 \in H$, we have*

$$\left|\mathbb{E}\, F\big(u(k;u_0,h+\eta)\big) - \int_H F(v)\mu(dv)\right| \leq C\big(1+\|u_0\|\big)e^{-\gamma k}, \quad k \geq 0. \tag{1.6}$$

We refer the reader to Section 2.1 for a more general result on uniqueness of a stationary measure and exponential mixing. The proof of the above theorem is based on a detailed study of controllability properties[(1)] of problem (1.1)–(1.3) (in which η plays the role of a control), a general criterion for mixing of Markov chains, and a result on the image of measures on a Hilbert space under finite-dimensional transformations; see Section 2.2 for more details.

Let us mention that the problem of ergodicity for the 2D Navier-Stokes system was studied intensively in the last twenty years. First results in this direction were established in [10, 22, 8, 6], and we refer the reader to the book [24] for further references and description of the methods used in various works. Most of the results established so far concern the situation in which the random force is non-degenerate in a set of determining modes of the problem. In the case when the equation is studied on the torus and the deterministic force is zero, it was proved in [14, 15] that the Navier-Stokes dynamics is exponentially mixing for any $\nu > 0$, provided that the noise is white in time and has a few non-zero Fourier modes as a function of x (thus, it is finite-dimensional in x, infinite-dimensional in time, and localised in the Fourier space). This result was extended to the case of 2D sphere in the paper [16], which also fixes an error in [14]. The main theorem stated above is valid for all $\nu > 0$ and, to the best of our knowledge, provides a first result on mixing properties for Navier-Stokes equations with a space-time localised noise.

In conclusion, let us mention that the results of this paper remain valid in the case when the noises act through the boundary of the domain. This situation will be addressed in a subsequent publication.

The paper is organised as follows. In Section 2, we formulate the main result of this paper on exponential mixing for the Navier-Stokes system with space-time localised noise, outline its proof, and discuss some examples. Section 3 is devoted to studying a control problem associated with the stochastic system in question. The details of proof of the main result are given in Section 4. The appendix gathers some auxiliary results used in the main text.

(1) Note, however, that we do not deal at all with the Gramian of the control problem in question, and the property we use may be called *squeezing by a finite-dimensional modification*.