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SHARP STRICHARTZ ESTIMATES FOR THE WAVE EQUATION ON A ROUGH BACKGROUND

BY JÉRÉMIE SZEFTTEL

ABSTRACT. – In this paper, we obtain sharp Strichartz estimates for solutions of the wave equation $\square_{\mathbf{g}}\phi = 0$ where \mathbf{g} is a rough Lorentzian metric on a 4 dimensional space-time \mathcal{M} . This is the last step of the proof of the bounded L^2 curvature conjecture proposed in [3], and solved by S. Klainerman, I. Rodnianski and the author in [7], which also relies on the sequence of papers [15] [16] [17] [18]. Obtaining such estimates is at the core of the low regularity well-posedness theory for quasilinear wave equations. The difficulty is intimately connected to the regularity of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ for a rough metric \mathbf{g} . In order to be consistent with the final goal of proving the bounded L^2 curvature conjecture, we prove Strichartz estimates for all admissible Strichartz pairs under minimal regularity assumptions on the solutions of the eikonal equation.

RÉSUMÉ. – Dans cet article, nous obtenons des estimations de Strichartz optimales pour les solutions de l'équation des ondes $\square_{\mathbf{g}}\phi = 0$ où \mathbf{g} est une métrique lorentzienne peu régulière sur un espace-temps \mathcal{M} de dimension 4. Il s'agit de la dernière étape de la preuve de la conjecture de courbure L^2 proposée dans [3], et résolue par S. Klainerman, I. Rodnianski et l'auteur dans [7], qui repose également sur la série d'articles [15] [16] [17] [18]. De telles estimations sont au cœur de la théorie de l'existence locale pour les équations d'ondes non linéaires en faible régularité. La difficulté est intimement liée à la régularité de l'équation eikonale $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ pour une métrique peu régulière \mathbf{g} . Avec pour but final la preuve de la conjecture de courbure L^2 , nous prouvons des estimations de Strichartz pour toutes les paires admissibles sous des hypothèses minimales de régularité pour l'équation eikonale.

1. Introduction

In this paper, we obtain sharp Strichartz estimates for solutions of the wave equation $\square_{\mathbf{g}}\phi = 0$ where \mathbf{g} is a rough Lorentzian metric on a 4 dimensional space-time \mathcal{M} . This is the last step of the proof of the bounded L^2 curvature conjecture proposed in [3], and solved by S. Klainerman, I. Rodnianski and the author in [7], which also relies on the sequence of papers [15] [16] [17] [18]. Obtaining such estimates is at the core of the low regularity well-posedness theory for quasilinear wave equations. The difficulty is intimately connected to

the regularity of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ for a rough metric \mathbf{g} . In order to be consistent with the final goal of proving the bounded L^2 curvature conjecture, we prove Strichartz estimates for all admissible Strichartz pairs under minimal regularity assumptions on the solutions of the eikonal equation.

Since we are ultimately interested in local well-posedness, it is enough to prove local in time Strichartz estimates. Also, it is natural to prove Strichartz estimates which are localized in frequency⁽¹⁾. Finally, an $L_t^\infty L_x^2$ type bound in the context of the bounded L^2 curvature conjecture follows from the analysis in [16] [18], so we will assume that such a bound holds in this paper⁽²⁾. Thus, we focus in this paper on the issue of proving local in time Strichartz estimates which are localized in frequency assuming an a priori $L_t^\infty L_x^2$ bound. In particular, this turns out to be sufficient for the proof of the bounded L^2 curvature conjecture.

We start by recalling the sharp Strichartz estimates for the standard wave equation on $(\mathbb{R}^{1+3}, \mathbf{m})$ where \mathbf{m} is the Minkowski metric. We consider ϕ solution of

$$(1.1) \quad \begin{cases} \square\phi = 0, (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \phi(0, \cdot) = \phi_0, \partial_t\phi(0, \cdot) = \phi_1, \end{cases}$$

where

$$\square = \square_{\mathbf{m}} = -\partial_t^2 + \Delta_x.$$

Let (p, q) such that $p, q \geq 2$, $q < +\infty$, and

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}.$$

Let r defined by

$$r = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}.$$

We call (p, q, r) an admissible pair. Then, the solution ϕ of (1.1) satisfies the following estimates, called Strichartz estimates [13] [14]

$$(1.2) \quad \|\phi\|_{L^p(\mathbb{R}^+, L^q(\mathbb{R}^3))} \lesssim \|\phi_0\|_{\dot{H}^r(\mathbb{R}^3)} + \|\phi_1\|_{\dot{H}^{r-1}(\mathbb{R}^3)}.$$

Strichartz estimates allow to obtain well-posedness results for nonlinear wave equations with less regularity for the Cauchy data (ϕ_0, ϕ_1) than what is typically possible by relying only on energy methods (see for example [8] in the context of semilinear wave equations). Therefore, as far as low regularity well-posedness theory for quasilinear wave equations is concerned, a considerable effort was put in trying to derive Strichartz estimates for the wave equation

$$(1.3) \quad \square_{\mathbf{g}}\phi = 0$$

⁽¹⁾ The standard proof of Strichartz estimates in the flat case proceeds in two steps (see for example [11]). First, one localizes in frequency using Littlewood-Paley theory. Then, one proves the corresponding Strichartz estimates localized in frequency.

⁽²⁾ The standard proof of Strichartz estimates in the flat case relies in particular on an interpolation argument between the $L_t^\infty L_x^2$ bound and a dispersive bound. The $L_t^\infty L_x^2$ bound is usually obtained by other methods - for the wave equation in Minkowski, it follows from the conservation of energy - so we will focus in this paper on the derivation of the dispersive estimate.

on a space-time $(\mathcal{M}, \mathbf{g})$ where \mathbf{g} has limited regularity, see [9], [2], [1], [19], [20], [4], [5], [10]. All these methods have in common a crucial and delicate analysis of the regularity of solutions u to the eikonal equation

$$\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0.$$

To illustrate the role played by the eikonal equation, let us first recall the plane wave representation of the standard wave equation. The solution ϕ of (1.1) is given by:

$$(1.4) \quad \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i(-t+x\cdot\omega)\lambda} \frac{1}{2} \left(\mathcal{F}\phi_0(\lambda\omega) + i \frac{\mathcal{F}\phi_1(\lambda\omega)}{\lambda} \right) \lambda^2 d\lambda d\omega \\ + \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i(t+x\cdot\omega)\lambda} \frac{1}{2} \left(\mathcal{F}\phi_0(\lambda\omega) - i \frac{\mathcal{F}\phi_1(\lambda\omega)}{\lambda} \right) \lambda^2 d\lambda d\omega,$$

where \mathcal{F} denotes the Fourier transform on \mathbb{R}^3 . The plane wave representation (1.4) is the sum of two half waves, and Strichartz estimates are derived for each half-wave separately with an identical proof so we may focus on the first half-wave which we rewrite under the form

$$(1.5) \quad \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i(-t+x\cdot\omega)\lambda} f(\lambda\omega) \lambda^2 d\lambda d\omega$$

where the function f on \mathbb{R}^3 is explicitly given in term of the Fourier transform of the initial data. Note that $-t + x \cdot \omega$ is a family of solutions to the eikonal equation in the Minkowski space-time depending on the extra parameter $\omega \in \mathbb{S}^2$. The natural generalization of (1.5) to the curved case is the following representation formula - also called parametrix

$$(1.6) \quad \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega) \lambda^2 d\lambda d\omega$$

where u is a family of solutions to the eikonal equation in the curved space-time $(\mathcal{M}, \mathbf{g})$ depending on the extra parameter $\omega \in \mathbb{S}^2$. Thus, our parametrix is a Fourier integral operator with a phase u satisfying the eikonal equation⁽³⁾.

Assume now that the space-time \mathcal{M} is foliated by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t . The estimate for the parametrix (1.6) corresponding to the Strichartz estimates of the flat case (1.2) is

$$(1.7) \quad \left\| \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega) \lambda^2 d\lambda d\omega \right\|_{L^p(\mathbb{R}^+, L^q(\Sigma_t))} \lesssim \|\lambda^r f\|_{L^2(\mathbb{R}^3)}.$$

Since we are ultimately interested in local well-posedness, it is enough to restrict the time interval to $[0, 1]$, which corresponds to local in time Strichartz estimates. Also, it is natural to prove Strichartz estimates which are localized in frequency (see footnote 1). Finally, an $L_t^\infty L_x^2$ type bound in the context of the bounded L^2 curvature conjecture follows from the analysis in [16] [18], so we will assume that such a bound holds in this paper. Thus we focus on proving Strichartz estimates on the time interval $[0, 1]$ for a parametrix localized in a dyadic shell for which an a priori $L_t^\infty L_x^2$ bound is assumed. Let $j \geq 0$, and let ψ a smooth function on \mathbb{R} supported in

$$\frac{1}{2} \leq \lambda \leq 2.$$

⁽³⁾ We refer to [16] [18] for a precise construction of a parametrix of the form (1.6) which generates any initial data of (1.3) and for its control in the context of the bounded L^2 curvature theorem of [7]