ISSN 0012-9593

quatrième série - tome 50 fascicule 1 janvier-février 2017

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition / *Publication*

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Tarifs

Europe : 519 €. Hors Europe : 548 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 50, 2017, p. 1 à 37

CHARACTERIZATIONS OF RECTIFIABLE METRIC MEASURE SPACES

BY David BATE AND Sean LI

ABSTRACT. – We characterize n-rectifiable metric measure spaces as those spaces that admit a countable Borel decomposition so that each piece has positive and finite n-densities and one of the following: is an n-dimensional Lipschitz differentiability space; has n-independent Alberti representations; satisfies David's condition for an n-dimensional chart. The key tool is an iterative grid construction which allows us to show that the image of a ball with a high density of curves from the Alberti representations under a chart map contains a large portion of a uniformly large ball and hence satisfies David's condition. This allows us to apply modified versions of previously known "biLipschitz pieces" results [8, 12, 10, 21] on the charts.

RÉSUMÉ. – Nous caractérisons les espaces métriques mesurés n-rectifiables comme étant les espaces qui admettent une décomposition borélienne dénombrable telle que chaque morceau admet une n-densité finie et strictement positive, et vérifie l'une des conditions suivantes : c'est un espace de Lipschitz-différentiabilité de dimension n ; il admet n représentations d'Alberti indépendantes ; il satisfait à la condition de David pour une carte de dimension n. L'outil essentiel est une construction de grille itérative qui nous permet de montrer que l'image par une application de carte d'une boule ayant une grande densité de courbes des représentations d'Alberti contient une grande proportion d'une boule de grand rayon, et donc vérifie la condition de David. Cela nous permet d'appliquer des versions modifiées de résultats connus concernant les « morceaux bilipschitz » [8, 12, 10, 21] sur les cartes.

1. Introduction

A metric measure space (X, d, μ) is said to be n-rectifiable if there exists a countable family of Lipschitz functions f_i defined on measurable subsets $A_i \subset \mathbb{R}^n$ such that $\mu(X \setminus \bigcup_{i=1}^{\infty} f(A_i)) = 0$ and $\mu \ll \mathcal{H}^n$. We say that a metric space is n-rectifiable if it is n-rectifiable when equipped with $\mathcal{H}^n$. Similarly to how rectifiable subsets of Euclidean space possess many nice properties akin to those of smooth manifolds, rectifiable metric measure spaces also satisfy many regularity properties. For these reasons, it is highly desirable to find general conditions that describe when a metric measure space is rectifiable.

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Such conditions have been difficult to find. Classically (that is, when the measure is defined on Euclidean space), this problem was first fully solved by Mattila [16] for Hausdorff measure and more generally by Preiss [18] for an arbitrary Radon measure. Both of these results consider the upper and lower n-densities of the measure defined by

$$\Theta^{*,n}(\mu;x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n} \qquad \text{and} \qquad \Theta^n_*(\mu;x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n}$$

respectively. When these two values agree, we denote the common value by $\Theta^n(\mu;x)$, the n-dimensional density of μ at x. Precisely, Mattila shows that, if $\Theta^n(\mathcal{H}^n \llcorner E;x) = 1$ for $\mathcal{H}^n$-a.e. $x \in E \subset \mathbb{R}^N$, then E is n-rectifiable. More generally, Preiss showed that if μ is a Radon measure on $\mathbb{R}^N$ for which $\Theta^n(\mu;x)$ exists and is positive and finite μ-a.e. then $(\mathbb{R}^N,\mu)$ is n-rectifiable.

In the metric setting, only partial answers are known. We first partially recall a theorem of Kirchheim [13] that will be fundamental to our characterization of rectifiable metric measure spaces.

THEOREM 1.1 (Kirchheim [13]). – *Let (X,d) be an n-rectifiable metric space of finite $\mathcal{H}^n$ measure. Then $\Theta^n(\mathcal{H}^n;x) = 1$ for $\mathcal{H}^n$ a.e. x.*

Conversely, Preiss and Tišer [19] showed that, for dimension 1, a lower Hausdorff density greater than (a number slightly less than) $3/4$ is sufficient to ensure 1-rectifiability.

In a different direction, but with a similar goal, the work of David and Semmes [8, 21, 10] found conditions under which a Lipschitz function defined on an Ahlfors n-regular space taking values in $\mathbb{R}^n$ is in fact biLipschitz. A metric measure space is said to be Ahlfors n-regular if there exist $C > 0$ such that

$$\frac{1}{C} r^n \leq \mu(B(x,r)) \leq C r^n$$

for each $x \in X$ and $r < \operatorname{diam}(X)$. One may consider this condition to be a quantitative version of the property that the upper and lower n-densities are positive and finite. David and Semmes showed that, if the image of every ball under a Lipschitz function contains a ball of comparable radius centred at the image of centre of the first ball (that is, the function is a Lipschitz quotient), then the function can be decomposed into biLipschitz pieces. In fact, they only require that the image contains most (in terms of measure) of a ball of comparable radius. We will discuss generalizations of this condition, now known as David's condition, below. In particular, our main tool of constructing rectifiable subsets of metric measure spaces will be a generalization of the theorems of David and Semmes that are applicable when the space only satisfies density estimates, rather than full Ahlfors regularity. See Theorem 5.3 for the statement of the generalization.

More recently, initiated by the striking work of Cheeger [3], there has been much activity in generalizing the classical theorem of Rademacher to metric measure spaces. Most of all, this departed from the existing generalizations of Rademacher's theorem, for example that of Pansu [17], by not requiring a group structure in the domain to define the derivative. Let $U \subset X$ be a Borel set and $\varphi\colon X \to \mathbb{R}^n$ Lipschitz. We say that (U,φ) form a chart of dimension n and that a function $f : X \to \mathbb{R}$ is differentiable at $x_0 \in U$ with respect to (U, φ) if there exists a *unique* $Df(x_0) \in \mathbb{R}^n$ (the derivative of f at x_0) so that

$$\lim_{X \ni x \to x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

A metric measure space (X, d, μ) is said to be a Lipschitz differentiability space if there exists a countable set of charts (U_i, φ_i) so that $X = \bigcup_i U_i$ and every real valued Lipschitz function defined on X is differentiable at almost every point of every chart. A Lipschitz differentiability space is said to be n-dimensional if every chart map is $\mathbb{R}^n$ valued.

Although the concept of a Lipschitz differentiability space is very general, it is known that some additional structure must exist—it must be possible to decompose the measure into an integral combination of 1-rectifiable measures known as Alberti representations. Define Γ to be the collection of biLipschitz functions γ defined on a compact subset of $\mathbb{R}$ taking values in X (known as curve fragments). We say that μ has an Alberti representation if there exists a probability measure $\mathbb{P}$ on Γ and measures $\mu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Image} \gamma$ such that

$$\mu(B) = \int_\Gamma \mu_\gamma(B) \, \mathrm{d}\mathbb{P}$$

for each Borel set $B \subset X$. Further, for a Lipschitz function $\varphi \colon X \to \mathbb{R}^n$, an Alberti representation is in the φ-direction of a cone $C \subset \mathbb{R}^n$ if

$$(\varphi \circ \gamma)'(t) \in C \setminus \{0\},$$

for $\mathbb{P}$ a.e. $\gamma \in \Gamma$ and μ_γ a.e. $t \in \operatorname{dom} \gamma$. Finally, we say that a collection of n Alberti representations are φ-independent if there exist linearly independent cones in $\mathbb{R}^n$ so that each Alberti representation is in the direction of a cone and no two Alberti representations are in the direction of the same cone. One of the main results of [1] is that any n-dimensional chart in a Lipschitz differentiability space has n independent Alberti representations.

There are known examples of Lipschitz differentiability spaces that are not groups (cf. [2, 5, 14]) and do not admit any rectifiable behaviour beyond the existence of Alberti representations. Indeed, Cheeger also showed that for many of these spaces to possess a biLipschitz embedding into any Euclidean space, the dimension of the chart must equal the Hausdorff dimension of the space, which is not generally true. (More generally, a theorem of Cheeger-Kleiner [4] proves the same result for a biLipschitz embedding into an RNP Banach space.) However, there are very natural relationships between rectifiability and differentiability. For example, Rademacher's theorem easily extends to rectifiable sets via composition of functions (this concept is at the heart of the relationship between Alberti representations and differentiability). Further, the proof of Kirchheim's theorem fundamentally relies on a version of Rademacher's theorem for metric space valued Lipschitz functions defined on Euclidean space. Once this is established, the outline is that of the classical case.

In this paper we give precise conditions when the notions of rectifiability and differentiability agree and hence obtain several characterizations of rectifiable metric measure spaces. Specifically, we prove the following theorem.

Theorem 1.2. – *A metric measure space (X, d, μ) is n-rectifiable (which we denote by Property (R)) if and only if there exist a countable collection of Borel sets $U_i \subset X$ with*