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MORRISON-KAWAMATA CONE CONJECTURE FOR HYPERKÄHLER MANIFOLDS

BY Ekaterina AMERIK AND Misha VERBITSKY

ABSTRACT. – Let M be a simple hyperkähler manifold, that is, a simply connected compact holomorphically symplectic manifold of Kähler type with $h^{2,0} = 1$. Assuming $b_2(M) \neq 5$, we prove that the group of holomorphic automorphisms of M acts on the set of faces of its Kähler cone with finitely many orbits. This statement is known as Morrison-Kawamata cone conjecture for hyperkähler manifolds. As an implication, we show that a hyperkähler manifold has only finitely many non-equivalent birational models. The proof is based on the following observation, proven with ergodic theory. Let M be a complete Riemannian manifold of dimension at least three, constant negative curvature and finite volume, and $\{S_i\}$ an infinite set of complete, locally geodesic hypersurfaces. Then the union of S_i is dense in M.

RÉSUMÉ. – Soit M une variété hyperkählérienne irréductible. En supposant $b_2(M) \neq 5$, nous montrons que le groupe d'automorphismes de M n'a qu'un nombre fini d'orbites sur l'ensemble des faces du cône de Kähler. Cet enoncé est une version de la conjecture de Morrison-Kawamata pour les variétés hyperkählériennes. Une conséquence en est la finitude du nombre des modèles birationnels pour une telle variété. La preuve s'appuie sur l'observation suivante, qui se démontre dans le cadre de la théorie ergodique : soient M une variété riemanienne complète de dimension au moins trois, de courbure constante négative et de volume fini, et $\{S_i\}$ un ensemble infini d'hypersurfaces localement géodésiques. Alors la réunion des S_i est dense dans M.

1. Introduction

1.1. Kähler cone and MBM classes

Let M be a hyperkähler manifold, that is, a compact, holomorphically symplectic Kähler manifold. We assume that $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$: the general case reduces to this by Bogomolov decomposition (2.3). Such hyperkähler manifolds are known as simple hyperkähler manifolds, or IHS (irreducible holomorphic symplectic) manifolds. The

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known examples of such manifolds are deformations of punctual Hilbert scheme of K3 surfaces, deformations of generalized Kummer varieties and two sporadic ones discovered by O'Grady. In [1] we gave a description of the Kähler cone of M in terms of a set of cohomology classes $S \subset H^2(M, \mathbb{Z})$ called **MBM classes** (2.14). This set depends only on the deformation type of M.

Recall that on the second cohomology of a hyperkähler manifold, there is an integral quadratic form q, called the Beauville-Bogomolov-Fujiki form (see Section 2 for details). This form is of signature $(+, -, \dots, -)$ on $H^{1,1}(M)$. Let $\operatorname{Pos} \subset H^{1,1}(M)$ be the positive cone, and $S(I)$ the set of all MBM classes which are of type (1,1) on M with its given complex structure I. Then the Kähler cone is a connected component of $\operatorname{Pos} \setminus S(I)^\perp$, where $S(I)^\perp$ is the union of the orthogonal complements to all $z \in S(I)$.

1.2. Morrison-Kawamata cone conjecture for hyperkähler manifolds

The Morrison-Kawamata cone conjecture for Calabi-Yau manifolds was stated in [27]. For K3 surfaces it was already known since mid-eighties by the work of Sterk [33]. Kawamata in [16] proved the relative version of the conjecture for Calabi-Yau threefolds admitting a holomorphic fibration over a positive-dimensional base.

In this paper, we concentrate on the following version of the cone conjecture (see Subsection 5.2 for its relation to the classical one, formulated for the ample cone of a projective variety).

DEFINITION 1.1. – *Let M be a compact, Kähler manifold,* $\operatorname{Kah} \subset H^{1,1}(M, \mathbb{R})$ *the Kähler cone, and* $\overline{\operatorname{Kah}}$ *its closure in* $H^{1,1}(M, \mathbb{R})$*, called* ***the nef cone****. A* ***face*** *of the Kähler cone is the intersection of the boundary of* $\overline{\operatorname{Kah}}$ *and a hyperplane* $V \subset H^{1,1}(M, \mathbb{R})$ *which has non-empty interior.*

CONJECTURE 1.2 (Morrison-Kawamata cone conjecture, Kähler version)
Let M be a Calabi-Yau manifold. Then the group $\operatorname{Aut}(M)$ *of biholomorphic automorphisms of M acts on the set of faces of* Kah *with finite number of orbits.*

The **original Morrison-Kawamata cone conjecture** is formulated for projective Calabi-Yau manifolds and has two versions: **the weak one** states that $\operatorname{Aut}(M)$ acts with finitely many orbits on the set of faces of the ample cone and **the strong one** states that $\operatorname{Aut}(M)$ has a finite polyhedral fundamental domain on the ample cone, or, more precisely, on the cone $\operatorname{Nef}^+(M)$ obtained from the ample cone by adding the "rational part" of its boundary (see [27, 34, 22] for details).

We shall be interested in the case when the manifold M is simple hyperkähler (that is, IHS). Our main purpose is to prove 1.2. Notice that the stronger version involving fundamental domains cannot be true in this Kähler setting, as for a very general IHS M the Kähler cone is equal to the positive cone whereas $\operatorname{Aut}(M)$ is trivial. However when M is projective IHS, the Kähler version of the conjecture implies almost immediately not only the weak, but also the strong original version (see Section 5).

In [1], we have shown that the Kähler version of the Morrison-Kawamata cone conjecture holds whenever the Beauville-Bogomolov square of primitive MBM classes is bounded. This is known to be the case for deformations of punctual Hilbert schemes of K3 surfaces and for deformations of generalized Kummer varieties.

The strong version of the cone conjecture for projective IHS under the boundedness assumption for primitive MBM classes has been proved by Markman and Yoshioka in [22]. In Section 5 we suggest a rapid alternative way to deduce this strong version from ours: the tools are Borel and Harish Chandra theorem on arithmetic subgroups and geometric finiteness results from hyperbolic geometry. To apply the first one, we have to suppose that the Picard number is at least three. The case of Picard number two has to be treated separately, but the argument is fairly easy. Thus it is the boundedness (in absolute value) of squares of primitive MBM classes which is at the heart of all versions of Morrison-Kawamata cone conjecture for IHS.

Let us also briefly mention that this conjecture has a birational version, proved for projective hyperkähler manifolds by E. Markman in [21] and generalized in [1] to the non-projective case. In this birational version, the nef cone is replaced by the birational nef cone (that is, the closure of the union of pullbacks of Kähler cones on birational models of M) and the group $\mathrm{Aut}(M)$ is replaced by the group of birational automorphisms $\mathrm{Bir}(M)$.

The key point of the proof of [1] is the observation that the orthogonal group $O(H^{1,1}_{\mathbb{Z}}(M), q)$ of the lattice $H^{1,1}_{\mathbb{Z}}(M) = H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, and therefore the **Hodge monodromy group** Γ_{Hdg} (see 2.12) which is a subgroup of finite index in $O(H^{1,1}_{\mathbb{Z}}(M), q)$, acts with finitely many orbits on the set of classes of fixed square $r \neq 0$. When the primitive MBM classes have bounded square, we conclude that the monodromy acts with finitely many orbits on the set of MBM classes. As those are precisely the classes whose orthogonal hyperplanes support the faces of the Kähler cone, it is not difficult to deduce that there are only finitely many, up to the action of the monodromy group, faces of the Kähler cone, and also finitely many oriented faces of the Kähler cone (an oriented face is a face together with the choice of normal direction). An element of the monodromy which sends a face F to a face F', with both orientations pointing towards the interior of the Kähler cone, must preserve the Kähler cone. On the other hand, Markman proved ([21], Theorem 1.3) that an element of the Hodge monodromy which preserves the Kähler cone must be induced by an automorphism, so that the cone conjecture follows.

1.3. Main results

The main point of the present paper is that the finiteness of the set of primitive MBM classes of type (1, 1), up to the monodromy action, can be obtained without the boundedness assumption on their Beauville-Bogomolov square.

Our main technical result is the following

THEOREM 1.3. – *Let L be a lattice of signature $(1, n)$ where $n \geqslant 3$, $V = L \otimes \mathbb{R}$. Let Γ be an arithmetic subgroup in $SO(1, n)$. Let $Y := \bigcup S_i$ be a Γ-invariant union of rational hyperplanes S_i orthogonal to negative vectors $z_i \in L$ in V. Then either Γ acts on $\{S_i\}$ with finitely many orbits, or Y is dense in the positive cone in V.*

Proof. – See 4.11. □