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OUTER AUTOMORPHISMS OF CLASSICAL ALGEBRAIC GROUPS

BY Anne QUÉGUINER-MATHIEU AND Jean-Pierre TIGNOL

Abstract. – The so-called Tits class associated to an adjoint absolutely almost simple algebraic group provides a cohomological obstruction for this group to admit an outer automorphism. If the group has inner type, this obstruction is the only one. In this paper, we prove this is not the case for classical groups of outer type, except for groups of type ${}^2\mathsf{A}_n$ with n even, or $n = 5$. More precisely, we prove a descent theorem for exponent 2 and degree 6 algebras with unitary involution, which shows that their automorphism groups have outer automorphisms. In all other relevant classical types, namely ${}^2\mathsf{A}_n$ with n odd, $n \geq 3$ and ${}^2\mathsf{D}_n$, we provide explicit examples where the Tits class obstruction vanishes, and yet the group does not have outer automorphisms. As a crucial tool, we use "generic" sums of algebras with involution.

Résumé. – À tout groupe algébrique absolument presque simple de type adjoint est associée une classe de cohomologie connue sous le nom de «classe de Tits» qui donne une obstruction à l'existence d'automorphismes extérieurs. Si le groupe est de type intérieur, il n'y a pas d'autre obstruction. Dans ce travail, nous montrons qu'il n'en va pas de même pour les groupes classiques de type extérieur, sauf pour les groupes de type ${}^2\mathsf{A}_n$ avec n pair ou $n = 5$. Plus précisément, nous établissons pour les algèbres à involution unitaire de degré 6 et d'exposant 2 un théorème de descente qui montre que les groupes d'automorphismes de ces algèbres ont des automorphismes extérieurs. Pour les types ${}^2\mathsf{A}_n$ avec n impair, $n \geq 3$, et les types ${}^2\mathsf{D}_n$, nous construisons des exemples explicites où l'obstruction donnée par la classe de Tits est nulle alors que le groupe ne possède pas d'automorphisme extérieur. Un outil crucial de nos constructions est la somme «générique» d'algèbres à involution.

1. Introduction

Every automorphism of an absolutely almost simple algebraic group scheme G of adjoint type over an arbitrary field F induces an automorphism of its Dynkin diagram Δ. Inner

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automorphisms of G act trivially on Δ, and there is an exact sequence of algebraic group schemes

$$1 \to G \to \mathbf{Aut}(G) \to \mathbf{Aut}(\Delta) \to 1, \tag{1}$$

see [2, exp. XXIV, 1.3, 3.6]. If G is split, the corresponding sequence of groups of rational points is exact and split, see [6, (25.16)], [11, §16.3]. Therefore, a split adjoint group G admits outer automorphisms if and only if its Dynkin diagram admits automorphisms, i.e., if G has type A_n with $n \geq 2$, D_n with $n \geq 3$ or E_6. Moreover, in all three cases, $\mathbf{Aut}(\Delta)(F)$ lifts to an isomorphic subgroup in $\mathbf{Aut}(G)(F)$. This property does not hold generally for nonsplit groups. For instance, if G is the connected component of the identity in the group scheme of automorphisms of a central simple F-algebra with quadratic pair (A, σ, f), then G has no outer automorphisms if A is not split by the quadratic étale F-algebra defined by the discriminant of the quadratic pair, see § 2.2 below. More generally, Garibaldi identified in [4, §2] a cohomological obstruction to the existence of outer automorphisms of an arbitrary absolutely almost simple algebraic group scheme G: the group $\mathbf{Aut}(\Delta)(F)$ acts on $H^2(F, C)$, where C is the center of the simply connected group scheme isogenous to G, and the Tits class $t_G \in H^2(F, C)$ is invariant under the action of the image of $\mathbf{Aut}(G)(F)$ in $\mathbf{Aut}(\Delta)(F)$. Therefore, automorphisms of Δ that do not leave t_G invariant do not lift to outer automorphisms of G. For adjoint or simply connected groups of inner type, Garibaldi showed in [4, §2] that this is the only obstruction to the lifting of automorphisms of Δ. As he explains in [4, Thm 11] this has interesting consequences in Galois cohomology. In a subsequent paper, Garibaldi-Petersson [5, Conjecture 1.1.2] conjectured that this Tits class obstruction is the only obstruction, also for adjoint or simply connected groups of outer type.

In this paper, we provide a complete answer to the question raised by Garibaldi and Petersson for groups of outer type A and D, leaving aside trialitarian groups (see the Appendix). Thus, in all the cases we consider, $\mathbf{Aut}(\Delta)(F)$ has order 2. Our main goal is to compare the following three conditions, listed from weaker to stronger:

(Out 1): The Tits class t_G is fixed under $\mathbf{Aut}(\Delta)(F)$;
(Out 2): G admits an outer automorphism defined over F;
(Out 3): G admits an outer automorphism of order 2 defined over F.

Under condition (Out 2), the sequence

$$1 \to G(F) \to \mathbf{Aut}(G)(F) \to \mathbf{Aut}(\Delta)(F) \to 1$$

is exact, and under condition (Out 3), it is split. In [4], Garibaldi proves that all three conditions are equivalent if G has inner type A or D (see Remarks 2.3 and 2.7). This is not the case for groups of outer type, and our main result is the following:

THEOREM 1.1. – *Let G be an absolutely almost simple adjoint or simply connected algebraic group scheme of type ${}^2\mathsf{A}_n$, with $n \geq 2$, or ${}^2\mathsf{D}_n$, with $n \geq 3$.*

(1) *If G has type ${}^2\mathsf{A}_n$, with n even, or ${}^2\mathsf{A}_5$, then conditions (Out 1), (Out 2) and (Out 3) are equivalent.*
(2) *In all the other types, there are examples of groups for which (Out 1) holds and (Out 2) does not hold, and examples of groups for which (Out 2) holds and (Out 3) does not hold.*

In other words, assertion (2) says there are examples where the condition on the Tits class is satisfied, and yet G does not have any outer automorphism, and examples where G has an outer automorphism, but no outer automorphism of order 2. In particular, this disproves Conjecture 1.1.2 in [5], and provides examples of simply connected absolutely simple algebraic group schemes G for which the Galois cohomology sequence

$$H^1(F, C) \to H^1(F, G) \to H^1(F, \mathbf{Aut}(G))$$

from [4, Thm 11(b)] (where C is the center of G) is not exact.

Every absolutely almost simple algebraic group scheme of adjoint type ${}^2\mathsf{A}_n$ over F is isomorphic to $\mathbf{PGU}(B, \tau) = \mathbf{Aut}_K(B, \tau)$ for some central simple algebra B of degree $n + 1$ over a separable quadratic field extension K of F with a K/F-unitary involution τ. As explained below in § 2.1, condition (Out 1) holds for the group $\mathbf{PGU}(B, \tau)$ if and only if B has exponent at most 2, and condition (Out 3) holds if and only if (B, τ) has a descent, i.e., $(B, \tau) = (B_0, \tau_0) \otimes_F (K, \iota)$ for some central simple F-algebra with F-linear involution (B_0, τ_0). For n even, Theorem 1.1(1) can be reformulated in a more precise form:

THEOREM 1.2. – *Let (B, τ) be a central simple algebra with unitary involution. If* $\deg B$ *is odd, then conditions (Out 1), (Out 2), and (Out 3) for* $\mathbf{PGU}(B, \tau)$ *are equivalent and hold if and only if B is split.*

The proof is easy: see Corollary 2.4.

Now, assume $G = \mathbf{PGU}(B, \tau)$ has type ${}^2\mathsf{A}_5$, i.e., B has degree 6. If the exponent of B is at most 2, then its index is at most 2. Therefore, Theorem 1.1(1) for such groups follows from the following descent theorem for algebras with unitary involution, proved in §4.1:

THEOREM 1.3. – *Let (B, τ) be a central simple algebra of degree at most* 6 *and index at most* 2*, with a K/F-unitary involution. There exists a central simple algebra with orthogonal involution (B_0, τ_0) over F, of the same index as B, such that $(B, \tau) = (B_0, \tau_0) \otimes (K, \iota)$, where ι is the unique nontrivial F-automorphism of K.*

It also follows from this theorem that assertion (1) does hold for groups of type ${}^2\mathsf{A}_3$ when the underlying algebra B has index at most 2; but this does not apply to all groups of type ${}^2\mathsf{A}_3$, since a degree 4 central simple algebra of exponent 2 can be of index 4. An example of a degree 4 and exponent 2 algebra with unitary involution that does not have a descent will be provided in § 3.3.2 below (see Remark 3.14).

As usual for classical groups, we use as a crucial tool their explicit description in terms of algebras with involution or quadratic pair. How conditions (Out 1), (Out 2) and (Out 3) translate into conditions on these algebraic structures is explained in § 2. Section 3 studies in more details the ${}^2\mathsf{D}_n$ case. In particular, we introduce our main tool for proving assertion (2) of Theorem 1.1, namely "generic" orthogonal sums of hermitian forms or involutions. In § 4, using the same kind of strategy, we prove Theorem 1.3, and complete the proof of Theorem 1.1 by producing examples of outer type ${}^2\mathsf{A}_n$.

We refer the reader to [6] for definitions and basic facts on central simple algebras, involutions, and quadratic pairs. Recall that if char $F \neq 2$, then for any quadratic pair (σ, f), σ is an involution of orthogonal type, and f is the map defined on the set $\mathrm{Sym}(A, \sigma)$ of σ-symmetric