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ON THE p-ADIC COHOMOLOGY OF THE LUBIN-TATE TOWER

BY PETER SCHOLZE
WITH AN APPENDIX BY MICHAEL RAPOPORT

ABSTRACT. – We prove a finiteness result for the p-adic cohomology of the Lubin-Tate tower. For any $n \geq 1$ and p-adic field F, this provides a canonical functor from admissible p-adic representations of $\mathrm{GL}_n(F)$ towards admissible p-adic representations of $\mathrm{Gal}_F \times D^\times$, where Gal_F is the absolute Galois group of F, and D/F is the central division algebra of invariant $1/n$.

Moreover, we verify a local-global-compatibility statement for this correspondence, and compatibility with the patching construction of Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin.

RÉSUMÉ. – Nous prouvons un résultat de finitude pour la cohomologie p-adique de la tour de Lubin-Tate. Pour tout $n \geq 1$ et corps p-adique F, cela fournit un functor canonique à partir de représentations p-adiques admissibles de $\mathrm{GL}_n(F)$ vers des représentations p-adiques admissibles de $\mathrm{Gal}_F \times D^\times$, où Gal_F est le groupe de Galois absolu de F, et D/F est l'algèbre à division centrale d'invariant $1/n$.

De plus, nous vérifions une compatibilité locale-globale pour cette correspondance, et une compatibilité avec le *patching* de Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin.

1. Introduction

The goal of this paper is to provide further evidence for the existence of a p-adic local Langlands correspondence, as was first envisioned by Breuil [4], and established for $\mathrm{GL}_2(\mathbb{Q}_p)$ by Colmez [14], Paskunas [30], and others. So far, little is known beyond $\mathrm{GL}_2(\mathbb{Q}_p)$, and work of Breuil-Paskunas [5], shows that already for $\mathrm{GL}_2(F)$, $F \neq \mathbb{Q}_p$, the situation is very difficult. There is a recent work of Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin, [7], that constructs *some* p-adic $\mathrm{GL}_n(F)$-representation starting from an n-dimensional representation of the absolute Galois group of a p-adic field F, for general n and F. Their construction is based on the patching construction of Taylor-Wiles, and is thus global in nature. Unfortunately, it is not clear that their construction gives a representation independent of the global situation.

In this paper, we work in the opposite direction. Namely, starting from a p-adic $\mathrm{GL}_n(F)$-representation π, we produce a representation $\mathcal{F}(\pi)$ of the absolute Galois

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group Gal_F, for any n and F, in a purely local way. Corollary 9.3 ensures that (for $n = 2$), composing the patching construction with our functor gives back the original Galois representation.

Actually, $F(\pi)$ also carries an admissible $D^\times$-action, where D/F is the central division algebra of invariant $1/n$. Thus, simultaneously, this indicates the existence of a p-adic Jacquet-Langlands correspondence relating p-adic $\mathrm{GL}_n(F)$ and $D^\times$-representations. Such a correspondence is not known already for $\mathrm{GL}_2(\mathbb{Q}_p)$, and its formalization remains mysterious, as the $D^\times$-representations are necessarily (modulo p) of infinite length. However, we do not pursue these questions here.

Let us now describe our results in more detail. Let $n \geq 1$ be an integer and $F/\mathbb{Q}_p$ a finite extension. Let $\mathcal{O} \subset F$ be the ring of integers, $\varpi \in \mathcal{O}$ a uniformizer, and let q be the cardinality of the residue field of F, which we identify with $\mathbb{F}_q$. Fix an algebraically closed extension k of $\mathbb{F}_q$, e.g., $\overline{\mathbb{F}}_q$. Let $\breve{F} = F \otimes_{W(\mathbb{F}_q)} W(k)$ be the completion of the unramified extension of F with residue field k. Let $\breve{\mathcal{O}} \subset \breve{F}$ be the ring of integers.

In this situation, one has the Lubin-Tate tower $(\mathcal{M}_{\mathrm{LT},K})_{K\subset \mathrm{GL}_n(F)}$, which is a tower of smooth rigid-analytic varieties $\mathcal{M}_{\mathrm{LT},K}$ over $\breve{F}$ parametrized by compact open subgroups K of $\mathrm{GL}_n(F)$, with finite étale transition maps. There is a compatible continuous action of $D^\times$ on all $\mathcal{M}_{\mathrm{LT},K}$, as well as an action of $\mathrm{GL}_n(F)$ on the tower, that is, $g \in \mathrm{GL}_n(F)$ induces an isomorphism between $\mathcal{M}_{\mathrm{LT},K}$ and $\mathcal{M}_{\mathrm{LT},g^{-1}Kg}$. There is the Gross-Hopkins period map, [26],

$$\pi_{\mathrm{GH}} : \mathcal{M}_{\mathrm{LT},K} \to \mathbb{P}^{n-1}_{\breve{F}},$$

compatible for varying K, which is an étale covering map of rigid-analytic varieties with fibers $\mathrm{GL}_n(F)/K$. It is also $D^\times$-equivariant if the right-hand side is correctly identified with the Brauer-Severi variety for D/F (which splits over $\breve{F}$). Moreover, there is a Weil descent datum on $\mathcal{M}_{\mathrm{LT},K}$, under which π_{GH} is equivariant for the above identification of $\mathbb{P}^{n-1}_{\breve{F}}$ with the Brauer-Severi variety of D/F.

It was first observed by Weinstein, cf. [35], that the inverse limit

$$\mathcal{M}_{\mathrm{LT},\infty} = \varprojlim_{K\subset \mathrm{GL}_n(F)} \mathcal{M}_{\mathrm{LT},K}$$

exists as a perfectoid space. The induced map

$$\pi_{\mathrm{GH}} : \mathcal{M}_{\mathrm{LT},\infty} \to \mathbb{P}^{n-1}_{\breve{F}}$$

is in a suitable sense a $\mathrm{GL}_n(F)$-torsor; however, it takes a little bit of effort to make this statement precise and we do not do so here. However, for any smooth $\mathrm{GL}_n(F)$-representation π on an $\mathbb{F}_p$-vector space, [(1)] one can construct a Weil-equivariant sheaf $\mathcal{F}_\pi$ on the étale site of the rigid space $\mathbb{P}^{n-1}_{\breve{F}}$. Our main theorem is the following:

Theorem 1.1. – *Let π be an admissible smooth $\mathrm{GL}_n(F)$-representation on an $\mathbb{F}_p$-vector space. The cohomology group*

$$H^i_{\text{ét}}(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$$

(1) One can also handle more general base rings, and we do so in the paper.

is independent of the choice of an algebraically closed complete extension C of $\breve{F}$, and vanishes for $i > 2(n-1)$. (2) *For all $i \geq 0$,*

$$H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathscr{F}_\pi)$$

is an admissible $D^\times$-representation, and the action of the Weil group W_F extends continuously to an action of the absolute Galois group Gal_F of F.

The proof of this theorem follows closely the proof of finiteness of $\mathbb{F}_p$-cohomology of proper (smooth) rigid spaces, [33]. In particular, it depends crucially on properness of $\mathbb{P}^{n-1}$, or more precisely, on properness of the image of π_{GH}. Unfortunately, it turns out that the Lubin-Tate case is essentially (up to products and changing the center) the only example of a Rapoport-Zink space with surjective period map. We refer to the Appendix by M. Rapoport for further discussion of this point. Thus, the methods of this paper do not shed light on other groups.

REMARK 1.2. – Intuitively, $H^*_{\text{ét}}(\mathbb{P}^{n-1}_C, \mathscr{F}_\pi)$ is the π-isotypic component of the cohomology of the Lubin-Tate tower, but the formulation is different for several reasons. First, the (usual or compactly supported) cohomology groups of $\mathscr{M}_{\mathrm{LT},0,C}$ or $\mathscr{M}_{\mathrm{LT},\infty,C}$ itself are not well-behaved, e.g., not admissible and not invariant under change of C, cf. work of Chojecki, [13]. Using lifts of Artin-Schreier covers one can check that already $H^1_{\text{ét}}(\mathbb{B}_C, \mathbb{F}_p)$ is infinite-dimensional and depends on C, where $\mathbb{B}_C$ denotes the closed unit disk over C. Second, taking the π-isotypic component is not an exact operation for $\mathbb{F}_p$-representations.

For the local-global-compatibility results, we have decided to work only with GL_2, as this leads to many technical simplifications; it is to be expected that many arguments can be adapted to GL_n if one uses Harris-Taylor type Shimura varieties, [25]. Fix a totally real field F and a place $\mathfrak{p}$ dividing p such that $F_\mathfrak{p}$ is the p-adic field considered previously. Moreover, fix an infinite place ∞_F of F. Let D_0 be a division algebra over F which is split at $\mathfrak{p}$ and is ramified at all infinite places. Let $G = D_0^\times$ be the algebraic group of units in D_0. Let D be the inner form of G which is split at ∞_F and ramified at $\mathfrak{p}$ (and unchanged at all other places), and denote by $D^\times$ the algebraic group of units of D. Fix a compact open subgroup $U^\mathfrak{p} \subset G(\mathbb{A}^\mathfrak{p}_{F,f}) \cong D^\times(\mathbb{A}^\mathfrak{p}_{F,f})$. For each $K \subset \mathrm{GL}_2(F_\mathfrak{p}) \cong G(F_\mathfrak{p})$, one has the space of algebraic automorphic forms

$$S(KU^\mathfrak{p}, \mathbb{F}_p) = C^0(G(F)\backslash G(\mathbb{A}_{F,f})/KU^\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p),$$

as well as the cohomology

$$H^1(\mathrm{Sh}_{K'U^\mathfrak{p},\mathbb{C}}, \mathbb{Q}_p/\mathbb{Z}_p)$$

of the Shimura curve $\mathrm{Sh}_{K'U^\mathfrak{p}}/F$ for D/F, for varying $K' \subset D_\mathfrak{p}^\times = D^\times(F_\mathfrak{p})$. These H^0- (resp. H^1-) groups are respectively the middle cohomology groups of the relevant Shimura varieties. Let

$$\pi = \varinjlim_K S(KU^\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p)$$

(2) As pointed out by the referee, the vanishing for $i > 2(n-1)$ follows already from a general result of Berkovich, [1, Theorem 2.5.1], cf. also [27, Corollary 2.8.3].