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***Abstract.*** — We study the structure possessed by the Goodwillie derivatives of a pointed homotopy functor of based topological spaces. These derivatives naturally form a bimodule over the operad consisting of the derivatives of the identity functor. We then use these bimodule structures to give a chain rule for higher derivatives in the calculus of functors, extending that of Klein and Rognes. This chain rule expresses the derivatives of $FG$ as a derived composition product of the derivatives of $F$ and $G$ over the derivatives of the identity.

There are two main ingredients in our proofs. Firstly, we construct new models for the Goodwillie derivatives of functors of spectra. These models allow for natural composition maps that yield operad and module structures. Then, we use a cosimplicial cobar construction to transfer this structure to functors of topological spaces. A form of Koszul duality for operads of spectra plays a key role in this.

***Résumé* (Opérades et règles de la chaîne pour le calcul fonctoriel).** — Nous étudions la structure des dérivées de Goodwillie d'un foncteur d'homotopie pointé d'espaces topologiques possédant une base. Ces dérivées forment, de manière naturelle, un bimodule au-dessus de l'opérade, celui des dérivées du foncteur identité. Nous utilisons ces structures de bimodule pour donner une règle de la chaîne pour les dérivées supérieures en calcul fonctoriel, étendant celle de Klein et Rognes. La règle de la chaîne exprime les dérivées de $FG$ en tant que produits de composition des dérivées de $F$ et de $G$ au-dessus des dérivées de l'identité.

Il y a deux ingrédients principaux dans nos preuves. Premièrement, nous construisons des nouveaux modèles pour les dérivées de Goodwillie des foncteurs de spectres. Ces modèles fournissent des applications de composition naturelles avec des structures de module et d'opérade. Ensuite, nous utilisons une construction de cobarre cosimplicielle pour porter cette structure aux foncteurs d'espaces topologiques. Une forme de dualité de Koszul pour les opérades de spectres joue un rôle-clé dans cette preuve.

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# INTRODUCTION

In a landmark series of papers, [**16**], [**17**] and [**18**], Goodwillie outlines his 'calculus of homotopy functors'. Let $F : \mathscr{C} \to \mathscr{D}$ (where $\mathscr{C}$ and $\mathscr{D}$ are each either $\mathsf{Top}_*$, the category of pointed topological spaces, or $\mathsf{Spec}$, the category of spectra) be a pointed homotopy functor. One of the things that Goodwillie does is associate with $F$ a sequence of spectra, which are called the *derivatives* of $F$. We denote these spectra by $\partial_1 F, \partial_2 F, \dots, \partial_n F, \dots$, or, collectively, by $\partial_* F$. Importantly, for each $n$ the spectrum $\partial_n F$ has a natural action of the symmetric group $\Sigma_n$. Thus, $\partial_* F$ is a *symmetric sequence* of spectra.

The importance of the derivatives of $F$ is that they contain substantial information about the homotopy type of $F$. Goodwillie's main construction in [**18**] defines a sequence of 'approximations' to $F$ together with natural transformations forming a so-called 'Taylor tower'. This tower takes the form

$$F \to \cdots \to P_n F \to P_{n-1} F \to \cdots \to P_0 F$$

with $P_n F$ being the universal '$n$-excisive' approximation to $F$. (A functor is *$n$-excisive* if it takes any $n+1$-dimensional cube with homotopy pushout squares for faces to a homotopy cartesian cube.) For 'analytic' $F$, this tower *converges* for sufficiently highly connected $X$, that is

$$F(X) \simeq \operatorname*{holim}_n P_n F(X).$$

In order to understand the functors $P_n F$ better, Goodwillie analyzes the fibre $D_n F$ of the map $P_n F \to P_{n-1} F$. This fibre is an '$n$-homogeneous' functor in an appropriate sense, and Goodwillie shows in [**18**] that $D_n F$ is determined by $\partial_n F$, via the following formula. If $F$ takes values in $\mathsf{Spec}$ then

$$D_n F(X) \simeq (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}.$$

If $F$ takes values in $\mathsf{Top}_*$ then one needs to prefix the right hand side with $\Omega^\infty$.

This paper investigates the question of what additional structure the collection $\partial_* F$ naturally possesses, beyond the symmetric group actions. The first example of such structure was given by the second author in [**9**]. There, he constructed an operad structure on the sequence $\partial_* I_{\mathsf{Top}_*}$, where $I_{\mathsf{Top}_*}$ is the identity functor on $\mathsf{Top}_*$. Our first main result says that if $F$ is a functor from $\mathsf{Top}_*$ to $\mathsf{Top}_*$, then $\partial_* F$ has the structure of a bimodule over the operad $\partial_* I_{\mathsf{Top}_*}$. (For functors either only from or to $\mathsf{Top}_*$, we get left or right module structures respectively.)

It turns out that these bimodule structures are exactly what is needed to write a 'chain rule' for the calculus of functors. By a chain rule we mean a formula for describing the derivatives of a composite functor $FG$ in terms of $\partial_*F$ and $\partial_*G$. Such a chain rule was first studied by Klein and Rognes, in **[27]**, who provided a complete answer to this question for first derivatives. In this paper we extend some of their work to higher derivatives, although with some restrictions. In particular, we only consider reduced functors (those with $F(*) \simeq *$) and only derivatives based at the trivial object $*$. (Klein and Rognes consider derivatives at a general base object.) Our result expresses $\partial_*(FG)$ as a derived 'composition product' of the $\partial_* I_{\mathsf{Top}_*}$-bimodule structures on $\partial_*F$ and $\partial_*G$.

The proofs of our main theorems are rather roundabout, but give us additional interesting results along the way. We first treat the case of functors from $\mathsf{Spec}$ to $\mathsf{Spec}$, and construct new models for the derivatives of such functors. Then, to pass from $\mathsf{Spec}$ to $\mathsf{Top}_*$, we rely heavily on the close connection between topological spaces and coalgebras over the cooperad $\Sigma^\infty\Omega^\infty$, and also on a form of 'Koszul duality' for operads in $\mathsf{Spec}$. Koszul duality for operads was first introduced by Ginzburg and Kapranov, in **[14]**, in the context of operads of chain complexes. Some of their ideas were extended to operads of spectra by the second author in **[9]**. In particular, it was shown there that the operad $\partial_* I_{\mathsf{Top}_*}$ plays the role of the Koszul dual of the commutative cooperad, and hence is a spectrum-level version of the Lie operad. In this paper, we give a deeper topological reason behind this observation. The commutative cooperad appears because it is equivalent to the derivatives of the comonad $\Sigma^\infty\Omega^\infty$. One of our main results is that the derivatives of $I_{\mathsf{Top}_*}$ and $\Sigma^\infty\Omega^\infty$ are related by this form of Koszul duality.

The module and bimodule structures on the derivatives of a general functor $F$ also arise via an extension of Koszul duality ideas to spectra. For example, we show that for $F : \mathsf{Top}_* \to \mathsf{Top}_*$, the derivatives of $F$ and $\Sigma^\infty F$ are related by a corresponding duality between left $\partial_* I_{\mathsf{Top}_*}$-modules and left $\partial_*(\Sigma^\infty\Omega^\infty)$-comodules.

It seems to us that this paper gives a satisfactory answer to one of the open-ended questions proposed in the introduction to **[9]**: is there a deeper connection between calculus of functors and the theory of operads? Yes, there is a deeper connection. It stems from two basic sources. The first is the fact that composition of functors is related to the composition product of symmetric sequences. The second is the relationship between $I_{\mathsf{Top}_*}$ and $\Sigma^\infty\Omega^\infty$, which translates, on the level of derivatives, to Koszul duality of operads.

We now proceed with a more precise statement of our main results.

**Our results.** — As we mentioned already, our results are stated in the language of operads and modules over them. The collection of derivatives of a functor $F$ (of either based spaces or spectra) forms a symmetric sequence of spectra, that is, a sequence

$$\partial_*F = (\partial_1 F, \partial_2 F, \dots)$$