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THE STABILIZATION OF THE FROBENIUS-HECKE TRACES ON THE INTERSECTION COHOMOLOGY OF ORTHOGONAL SHIMURA VARIETIES

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THE STABILIZATION OF THE FROBENIUS-HECKE TRACES ON THE INTERSECTION COHOMOLOGY OF ORTHOGONAL SHIMURA VARIETIES

by

Yihang Zhu

Abstract. – We study Shimura varieties associated with special orthogonal groups over the field of rational numbers. We prove a version of Morel's formula for the Frobenius-Hecke traces on the intersection cohomology of the Baily-Borel compactification. Our main result is the stabilization of this formula. As an application, we compute the Hasse-Weil zeta function of the intersection cohomology in some special cases, using the recent work of Arthur and Taïbi on the endoscopic classification of automorphic representations of special orthogonal groups.

***Résumé.* (La stabilisation des traces de Frobenius-Hecke sur la cohomologie d'intersection des variétés des Shimura orthogonales).** – Nous étudions les variétés de Shimura associées à des groupes spéciaux orthogonaux sur le corps des nombres rationnels. Nous prouvons une version de la formule de Morel pour les traces de Frobenius-Hecke sur la cohomologie d'intersection de la compactification de Baily-Borel. Notre résultat principal est la stabilisation de cette formule. Comme application, nous calculons la fonction zêta de Hasse-Weil de la cohomologie d'intersection dans certains cas particuliers, en utilisant les travaux récents d'Arthur et Taïbi sur la classification endoscopique des représentations automorphes des groupes spéciaux orthogonaux.

CONTENTS

其始也，
皆收視反聽，
耽思傍訊，
精騖八極，
心遊萬仞。
其致也，
情曈曨而彌鮮，
物昭晰而互進。
陸機《文賦》

In the beginning,
All external vision and sound are suspended,
Perpetual thought itself gropes in time and space;
Then, the spirit at full gallop reaches the eight limits of the cosmos,
And the mind, self-buoyant, will ever soar to new insurmountable heights.
When the search succeeds,
Feeling, at first but a glimmer, will gradually gather into full luminosity,
Whence all objects thus lit up glow as if each the other's light reflects. [1]

Excerpt from *Essay on Literature* by LU Ji (261–303 AD)

1. Translated from Chinese by Chen Shixiang.

INTRODUCTION

Inspired by the early works of Eichler, Shimura, Kuga, Sato, and Ihara, the ongoing study of expressing Hasse-Weil zeta functions of Shimura varieties through automorphic L-functions remains a focal point within the Langlands program. Langlands approached this problem by proposing a comparison of the Frobenius-Hecke traces on the cohomology of Shimura varieties with the stable Arthur-Selberg trace formulas, as detailed in [**71, 72, 73**]. Kottwitz further formalized these ideas into precise conjectures [**60, 62**]. In this paper, we confirm a version of Kottwitz's conjecture specifically for the intersection cohomology of orthogonal Shimura varieties.

The conjectures

Let $(G, \mathcal{X})$ be a Shimura datum with reflex field E. For each sufficiently small compact open subgroup $K \subset G(\mathbb{A}_f)$, we have the Shimura variety

$$\mathrm{Sh}_K = \mathrm{Sh}_K(G, \mathcal{X}),$$

which is a smooth quasi-projective algebraic variety over E. Let $\overline{\mathrm{Sh}_K}$ be the Baily-Borel compactification of Sh_K. Let $\mathbf{IH}^*$ be the intersection cohomology of $\overline{\mathrm{Sh}_K} \otimes_E \overline{E}$ with $\overline{\mathbb{Q}}_\ell$-coefficients. (More generally, a non-trivial "automorphic" coefficient system is allowed, which we ignore in the introduction.) Let p be a hyperspecial prime for K, i.e., $K = K_p K^p$ with $K_p \subset G(\mathbb{Q}_p)$ a hyperspecial subgroup and $K^p \subset G(\mathbb{A}_f^p)$ a compact open subgroup. (Here $\mathbb{A}_f^p$ denotes the finite adeles away from p.) Assume that $p \neq \ell$. On $\mathbf{IH}^*$, we have commuting actions of $\mathrm{Gal}(\overline{E}/E)$ and the Hecke algebra $\mathcal{H}(G(\mathbb{A}_f^p) /\!/ K^p)_{\overline{\mathbb{Q}}_\ell}$ consisting of the $\overline{\mathbb{Q}}_\ell$-valued smooth compactly supported K^p-bi-invariant distributions on $G(\mathbb{A}_f^p)$. Fix $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p) /\!/ K^p)_{\overline{\mathbb{Q}}_\ell}$, and let $\Phi = \Phi_\mathfrak{p}$ be a geometric Frobenius at a place $\mathfrak{p}$ of E above p. Let $a \in \mathbb{Z}_{\geq 1}$.

***Conjecture 1* (Kottwitz, see [60, §10]).** — *The action of* $\mathrm{Gal}(\overline{E}/E)$ *on* $\mathbf{IH}^*$ *is unramified at* $\mathfrak{p}$, *and under simplifying assumptions of a group-theoretic nature, we have*

$$\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^a \mid \mathbf{IH}^k) = \sum_H \iota(G,H) ST^H(f^H). \qquad (0.1)$$

On the right, H *runs through the isomorphism classes of elliptic endoscopic data of* G. *For each* H, $ST^H(\cdot)$ *is the geometric side of the stable trace formula for* H, *and* f^H *is a function on* $H(\mathbb{A})$ *determined by the Shimura datum,* $f^{p,\infty}$, *and* a.

In addition, Kottwitz also formulated the following conjecture for the compact support cohomology $\mathbf{H}_c^*$ of $\mathrm{Sh}_K \otimes_E \overline{E}$.

***Conjecture 2* (Kottwitz, see [60, §7]).** — *The action of* $\mathrm{Gal}(\overline{E}/E)$ *on* $\mathbf{H}_c^*$ *is unramified at* $\mathfrak{p}$*, and under simplifying assumptions we have*

$$\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^a \mid \mathbf{H}_c^k) = \sum_H \iota(G,H) ST_e^H(f^H). \tag{0.2}$$

Here H *and* f^H *are the same as in Conjecture 1, while* $ST_e^H(\cdot)$ *is the elliptic part of the geometric side of the stable trace formula for* H*.*

The main result

Let (V,q) be a quadratic space over $\mathbb{Q}$ of signature $(n,2)$, where $n \geq 3$. We assume that V has a 2-dimensional totally isotropic subspace, which is automatic if $n \geq 5$. Let $G = \mathrm{SO}(V,q)$. We have a natural Shimura datum $(G, \mathcal{X})$, where $\mathcal{X}$ can be identified with the set of oriented negative definite planes in $V_{\mathbb{R}}$. This Shimura datum is of abelian type (but not of Hodge type). The associated Shimura varieties are called *orthogonal Shimura varieties*. They are n-dimensional varieties over the reflex field $\mathbb{Q}$.

***Theorem 1* (Corollary 8.17.5).** — *Conjecture 1 is true for the orthogonal Shimura varieties associate to* (V,q)*, for almost all primes* p *and for all sufficiently large* a*.*

We refer the reader to the statements of Theorem 1.8.4 and Corollary 8.17.5 for the precise meaning of "almost all primes p". Here we just mention that the set of primes to be excluded should depend on a fixed element f^∞ of the "full" Hecke algebra $\mathcal{H}(G(\mathbb{A}_f) /\!/ K)_{\overline{\mathbb{Q}}_\ell}$, whereas $f^{p,\infty}$ in (0.1) should be the component of f^∞ away from p, after p has been chosen.

Some remarks

From a group-theoretic point of view, both sides of (0.2) are less complicated compared to (0.1). In fact, the RHS of (0.2) has an elementary definition in terms of stable orbital integrals. For the LHS of (0.2), Kottwitz computed it for PEL Shimura varieties of type A or C in **[62]** by counting (virtual) abelian varieties with additional structures over finite fields and using the Grothendieck-Lefschetz-Verdier trace formula. He obtained:

$$\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^a \mid \mathbf{H}_c^k) = \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta) O_\gamma(f^{p,\infty}) \mathrm{TO}_\delta(\phi) \operatorname{Tr}(\gamma_0 \mid \mathbb{V}). \tag{0.3}$$

We do not explain the terms on the RHS in detail here, but only mention that they are group-theoretic in nature and include orbital integrals $O_\gamma(\cdot)$ and twisted orbital integrals $\mathrm{TO}_\delta(\cdot)$. In **[60]**, Kottwitz conjectured that (0.3) should hold for general Shimura varieties (at least under some simplifying assumptions of a group-theoretic nature). In the same paper Kottwitz *stabilized* the RHS of (0.3), namely he found [(2)] the functions f^H such that the RHS of (0.3) is equal to the RHS of (0.2). In [**51**], both the formula (0.3) and the stabilization step are generalized to arbitrary Shimura varieties of abelian type, and Conjecture 2 is proved for these varieties.

One should view Conjecture 1 as one step forward from Conjecture 2. From a spectral perspective, it is ST^H rather than ST^H_e that sees the "whole picture". More specifically, ST^H has a spectral expansion, from which one can eventually make a link to automorphic representations. By contrast it is unclear how ST^H_e can be directly related to spectral information in general.

We also mention that the expectation that the intersection cohomology is the correct cohomology to insert in (0.1) is motivated by Zucker's conjecture and Arthur's work on L^2-cohomology, among other things. We refer the reader to [**89**] for a more detailed discussion on these motivations.

Application: the Hasse-Weil zeta functions

In [**60**], Kottwitz showed that one can combine Conjecture 1 with the conjectural framework of Arthur parameters and Arthur's multiplicity conjectures to infer a description of the Galois-Hecke module $\mathbf{IH}^*$, and in particular a formula for the Hasse-Weil zeta function associated to $\mathbf{IH}^*$.

Currently some of these premises related to Arthur's conjectures have been established in special cases. Most notably, Arthur [**12**] has established the multiplicity conjectures for quasi-split classical groups. [(3)] In fact, our interest in delving into special orthogonal groups within this paper is driven by a desire to connect with Arthur's work. This intentional decision distinguishes our focus from similar groups such as GSpin, whose Shimura varieties display a relative simplicity in various aspects, for instance, being of Hodge type.

Unfortunately, when the rank is large the special orthogonal groups that have Shimura varieties cannot be quasi-split even over $\mathbb{R}$, because of the signature $(n, 2)$ condition. Arthur's work has been generalized to limited cases of inner forms by Taïbi [**113**] (building on earlier work of Kaletha [**47, 46**] and Arancibia-Moeglin-Renard [**5**], among others). We combine Theorem 1 with Arthur's and Taïbi's work to obtain the following theorem. Here we state it only for odd n for simplicity.

***Theorem 2* (Theorem 9.7.5, Remark 9.7.6). —** *Assume that n is odd, and that $G = \mathrm{SO}(V, q)$ is quasi-split at all finite places. For any finite set S of prime numbers, let $\zeta^S(\mathbf{IH}^*, s)$ be the S-partial Hasse-Weil zeta function associated to $\mathbf{IH}^*$. When*

2. The construction of f^H relies on the Langlands-Shelstad Transfer Conjecture and the Fundamental Lemma, which were unproven at the time of [**60**]. They are now theorems thanks to the work of numerous mathematicians, most notably Ngô and Waldspurger.

3. The results in [**12**] are contingent on the release of several upcoming papers, including the reference [A25], which have not appeared as of the time of writing.

S is sufficiently large, we have

$$\log \zeta^S(\mathbf{IH}^*, s) = \sum_{\psi} \sum_{\pi^\infty} \sum_{\nu} \dim(\pi^\infty)^K m(\pi^\infty, \psi, \nu)(-1)^n \nu(s_\psi) \log L^S(\mathcal{M}(\psi, \nu), s).$$

Here ψ runs through a certain set of Arthur's substitutes of global Arthur parameters, π^∞ runs through the away-from-∞ global packet of ψ, and ν runs through characters of the centralizer group of ψ (which is finite abelian). The three-fold summation is over a finite range. The numbers $m(\pi^\infty, \psi, \nu) \in \{0, 1\}$ and $\nu(s_\psi) \in \{\pm 1\}$ are defined in terms of constructions in **[12]** *and* **[113]**. *The term $L^S(\mathcal{M}(\psi, \nu), s)$ is a finite product of S-partial standard automorphic L-functions for general linear groups (with some shifting in the variable s), and hence has meromorphic continuation to $\mathbb{C}$. In particular, the above formula implies that $\zeta^S(\mathbf{IH}^*, s)$ has meromorphic continuation to $\mathbb{C}$.*

In the proof of Theorem 2, one crucial ingredient is a relatively simple formula for $ST^H(f^H)$ when the test function f^H is *stable cuspidal* at the real place; see Hypothesis 9.1.2. This formula follows from Kottwitz's stabilization of the L^2 Lefschetz number formula in his unpublished notes, and is also used in Morel's work **[90, 91]**. A self-contained proof of this formula for $ST^H(f^H)$, from a different point of view, is given in a recent paper by Z. Peng **[93]**.

We also prove a refinement of Theorem 2 concerning the decomposition of $\mathbf{IH}^*$ in the Grothendieck group of Galois-Hecke modules, under the same assumption on G. When n is odd (as well as in some cases when n is even), we express $\mathbf{IH}^*$ in terms of the known Galois representations associated to regular algebraic cuspidal automorphic representations of general linear groups, with multiplicities given in a similar way as the multiplicities in Theorem 2. See Theorem 9.8.5, Corollary 9.8.8, and Corollary 9.8.10. When n is even, both the computation of the partial Hasse-Weil zeta function and the decomposition of $\mathbf{IH}^*$ proved in this paper are weaker than the conjectures in **[60]**, in that a certain ambiguity up to outer automorphism is constantly present. This is due to the extra ambiguity in the endoscopic classification of representations for even special orthogonal groups in **[12]** and **[113]**, which seems intrinsic to the methods therein.

As a byproduct of our refinement of Theorem 2, we prove that if an Arthur parameter ψ contributes to $\mathbf{IH}^*$, then the cuspidal automorphic representations of general linear groups that constitute ψ all satisfy the Ramanujan-Petersson conjecture at almost all primes. These representations need not be regular algebraic, in which case the conjecture was previously known. See Theorem 9.8.5 (3) and Remark 9.8.6.

Reduction to the stabilization of the boundary terms

We now discuss the structure of the proof of Theorem 1. For some period of time, the study of the LHS of (0.1) had been restricted to sporadic low dimensional cases; see for instance [78]. The essential tools for treating arbitrary dimensions were developed by Morel [87, 88] (cf. [89]), who went on to prove Conjecture 1 for some unitary similitude Shimura varieties and the Siegel modular varieties of arbitrary dimensions in [90] and [91] respectively. We use Morel's work to obtain the following result for the orthogonal Shimura varieties associated to (V, q). We fix a minimal parabolic subgroup of $G = \mathrm{SO}(V, q)$ and fix a Levi component of it. Thus we get a notion of standard parabolic subgroups and standard Levi subgroups of G.

***Theorem 3* (Theorem 1.8.4). —** *For almost all primes p, we have*

$$\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^j \mid \mathbf{IH}^k) = \sum_M \operatorname{Tr}_M, \tag{0.4}$$

where M runs through the standard Levi subgroups of G.

Let us roughly describe the terms Tr_M. For $M = G$, we have

$$\operatorname{Tr}_G = \sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^j \mid \mathbf{H}_c^k),$$

where $\mathbf{H}_c^k$ is the compact support cohomology of $\mathrm{Sh}_{K,\overline{\mathbb{Q}}}$. For a proper M, the term Tr_M is a more complicated mixture of the following ingredients.

- The analogue of $\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^j \mid \mathbf{H}_c^k)$ for a boundary stratum in $\overline{\mathrm{Sh}_K}$. In another words, an enumeration of points on the stratum fixed under certain Frobenius-Hecke operators.
- The topological fixed point formula of Goresky-Kottwitz-MacPherson as in [34], for the trace of a Hecke operator on the compact support cohomology of a certain locally symmetric space.
- *Kostant-Weyl terms.* By this we mean characters for certain algebraic sub-representations of M_P inside
$$\mathbf{H}^*(\operatorname{Lie} N_P, \mathbb{V}),$$
where P is a standard parabolic subgroup of G containing M, and $P = M_P N_P$ is the standard Levi decomposition. These sub-representations are defined by certain truncations of weights, and can be understood in terms Kostant's theorem [53] describing $\mathbf{H}^*(\operatorname{Lie} N_P, \mathbb{V})$.

As we have already mentioned, in [51] the term Tr_G is computed and stabilized for all Shimura varieties of abelian type. Thus Tr_G is known to be equal to the RHS of (0.2). In view of this, Theorem 1 follows from Theorem 3 and the following result, which may be viewed as the "stabilization of the boundary terms".

***Theorem 4* (Theorem 8.17.2). —** *We have*

$$\sum_{M \subsetneq G} \operatorname{Tr}_M = \sum_H \iota(G, H)[ST^H(f^H) - ST_e^H(f^H)]. \tag{0.5}$$

Stabilization of the boundary terms

The method for proving Theorem 4 is by calculating the two sides of (0.5) and matching the explicit expressions. To calculate the RHS, we use Kottwitz's formula in his unpublished notes, as mentioned below Theorem 2. According to this formula (to be recalled in §8.3), we have an expansion of the form

$$ST^H(f^H) - ST^H_e(f^H) = \sum_{M' \neq H} ST^H_{M'}(f^H),$$

where M' runs through standard proper Levi subgroups of H, and each term $ST^H_{M'}(\cdot)$ has a relatively simple expression.

Roughly speaking, we label the pairs (H, M') appearing in the above summation by either a standard proper Levi subgroup M of G or the symbol $\emptyset$. We write $(H, M') \sim M$, or $(H, M') \sim \emptyset$. In order to prove Theorem 4, we need to show

$$\mathrm{Tr}_M = \sum_{(H,M') \sim M} ST^H_{M'}(f^H), \tag{0.6}$$

where M is either a standard proper Levi subgroup of G or the symbol $\emptyset$, and we define $\mathrm{Tr}_\emptyset$ to be 0. The proof of (0.6) involves the following ingredients.

(i) Fixed point formula for a boundary stratum. — We need a formula that enumerates points on a boundary stratum fixed under a Frobenius-Hecke operator, of a form similar to (0.3). The boundary stratum in question is (a finite quotient of) either a modular curve or a zero-dimensional Shimura variety, so such a formula is essentially a classical result. However, the zero-dimensional case causes some extra complication. We will come back to this technical point later in the introduction.

(ii) Archimedean comparison. — We need a series of identities between the archimedean contributions to the two sides of (0.6). These are identities between terms of two different natures, namely discrete series character values (which appear on the RHS of (0.6)) and Kostant-Weyl terms (which appear on the LHS of (0.6); see the discussion below Theorem 3). We establish such identities by explicit computation. On the discrete series side, we use formulas due to Harish-Chandra [**39**] and Herb [**42**]. On the Kostant-Weyl side, we use Kostant's theorem [**53**] and the Weyl character formula.

We point out that a priori it is not clear which identities between the archimedean contributions would eventually lead to the proof of (0.6). Finding the correct forms of the archimedean identities seems to be a harder task than proving them. It would be desirable to have a more conceptual understanding of how the archimedean comparison should be woven into the proof of (0.6) in general.

(iii) Computation at p. — We need to compute the p-adic contributions to the two sides of (0.6) explicitly. A priori there are more p-adic terms on the RHS than the

LHS. We will need to prove, among other things, that the extra terms eventually cancel each other.

This finishes our discussion on the structure of the proof of Theorem 1. Next we highlight three new features in the proof which did not show up in Morel's work [**91, 90**] for symplectic similitude and unitary similitude groups.

Arithmetic feature: Shimura varieties of abelian type

The orthogonal Shimura varieties are of abelian type and not of PEL type. In this paper we take as a black box the main result of [**51**] that proves Conjecture 2 for these Shimura varieties. In Morel's work, the Shimura varieties are of PEL type, and for them Conjecture 2 was already proved by Kottwitz.

The reason that Theorem 1 is proved only for primes outside an unspecified finite set is also due to a certain lack of understanding of Shimura varieties of abelian type. Ideally one would like to prove the theorem for all hyperspecial primes p, but a prerequisite for that would be a robust theory of integral models of the Baily-Borel and toroidal compactifications. Such a theory has been established by Madapusi Pera [**84**] in the case of Hodge type. For the Baily-Borel compactifications alone, a "crude" construction of the integral models in the case of abelian type has been given by Lan-Stroh [**70**]. However, for the above-mentioned purpose the integral models of toroidal compactifications are equally important, and this is currently unavailable beyond the case of Hodge type. [4]

All the difficulty about integral models of compactifications can be circumvented at the cost of excluding an unspecified finite set of primes, and this is the point of view taken in this paper. We refer the reader to §3.1 for a more detailed discussion.

Geometric feature: zero-dimensional boundary strata as quotients of Shimura varieties

In general, the boundary strata of the Baily-Borel compactification are naturally isomorphic to finite quotients of Shimura varieties at certain natural levels. Often these quotients are isomorphic to genuine Shimura varieties. However this is not true for the zero-dimensional boundary strata in the present case. From a group-theoretic point of view, this issue corresponds to the fact that the orthogonal Shimura datum does not satisfy Morel's axioms in [**90**, Chap. 1]. As a result, in the proof of Theorem 3 we need to modify the axiomatic approach in *loc. cit.*, and the terms Tr_M in (0.4) are also given by formulas that are slightly different from those in [**90, 91**].

4. Added in proof: Peihang Wu's recent thesis contains results on integral models of toroidal compactifications of Shimura varieties of abelian type.

Endoscopic-theoretic feature: normalizing transfer factors

In the proof of (0.6), signs are utterly important. One source of signs is the difference between the normalizations of transfer factors at the real place. The necessity of computing these signs was not emphasized in [**90, 91**]. For the orthogonal Shimura varieties, these signs form a delicate pattern.

To understand these signs we need to compare the normalization $\Delta_{j,B}$ introduced in [**60**, §7], and the Whittaker normalization. Here we explicitly fix $G_{\mathbb{R}}$ as a pure inner form of its quasi-split inner form $G^*_{\mathbb{R}}$ and fix a Whittaker datum for $G^*_{\mathbb{R}}$, so the Whittaker normalizations for the transfer factors between $G_{\mathbb{R}}$ and its endoscopic groups can be defined. The normalization $\Delta_{j,B}$ naturally shows up in the description of the archimedean component of f^H. To compare these two normalizations, we compare the corresponding spectral transfer factors that appear in the endoscopic character relations and compute the sign between them.

Extra complication arises when $G^*_{\mathbb{R}}$ has more than one equivalence class of Whittaker data. This happens if and only if $\dim V$ is divisible by 4, when there are precisely two equivalence classes. In this case, we need to study how the two (different) Whittaker normalizations relate to the explicit formulas of Waldspurger [**121**], the latter having the merit of being easier to keep track of when passing to Levi subgroups. In this direction we prove Theorem 6.3.11, which may be of independent interest in representation theory.