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SYMMETRIES OF THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. — Symmetries of the defocusing nonlinear Schrödinger equation are expressed in action-angle coordinates and characterized in terms of the periodic and Dirichlet spectrum of the associated Zakharov-Shabat system. Application: proof of the conjecture that the periodic spectrum $\cdots < \lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^- \leq \cdots$ of a Zakharov-Shabat operator is symmetric, *i.e.* $\lambda_k^\pm = -\lambda_{-k}^\mp$ for all k, if and only if the sequence $(\gamma_k)_{k\in\mathbb{Z}}$ of gap lengths, $\gamma_k := \lambda_k^+ - \lambda_k^-$, is symmetric with respect to $k = 0$.

Résumé (*Symétries de l'équation de Schrödinger non linéaire*). — Les symétries de l'équation de Schrödinger nonlinéaire sont exprimées dans les variables action-angles et caractérisées à l'aide du spectre périodique et du spectre de Dirichlet du système de Zakharov-Shabat associé. Comme application, nous démontrons la conjecture suivante : le spectre périodique $\cdots < \lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^- \leq \cdots$ de l'opérateur de Zakharov-Shabat est symétrique, *i.e.* $\lambda_k^\pm = -\lambda_{-k}^\mp$ pour tout k, si et seulement si la suite $(\gamma_k)_{k\in\mathbb{Z}}$ des longueurs des intervalles d'instabilité, $\gamma_k := \lambda_k^+ - \lambda_k^-$, est symétrique par rapport à $k = 0$.

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1. Introduction

The defocusing nonlinear Schrödinger equation NLS on the circle

$$i\,\partial_t\varphi = -\partial_x^2\varphi + 2|\varphi|^2\varphi \tag{1.1}$$

can be viewed as a completely integrable Hamiltonian system of infinite dimension. Indeed, on the phase $L^2(S^1;\mathbb{C})$, introduce the Poisson bracket

$$\{F,G\} := i\int_{S^1}\Big(\frac{\partial F}{\partial\varphi(x)}\cdot\frac{\partial G}{\partial\overline{\varphi}(x)} - \frac{\partial F}{\partial\overline{\varphi}(x)}\cdot\frac{\partial G}{\partial\varphi(x)}\Big)\,\mathrm{d}x.$$

Equation (1.1) can then be written in Hamiltonian form as follows

$$\frac{\partial\varphi}{\partial t} = \{\mathcal{H},\varphi\} = -i\,\frac{\partial\mathcal{H}}{\partial\overline{\varphi}},\quad \frac{\partial\overline{\varphi}}{\partial t} = \{\mathcal{H},\overline{\varphi}\} = i\,\frac{\partial\mathcal{H}}{\partial\varphi},$$

where the Hamiltonian $\mathcal{H}$ is given by (*cf.* [2])

$$\mathcal{H}(\varphi) := \int_{S^1}\Big(\Big|\frac{\partial\varphi}{\partial x}\Big|^2 + |\varphi|^4\Big)\,\mathrm{d}x.$$

Consider the following symmetry operators, acting on $L^2(S^1;\mathbb{C})$,

$$\mathcal{S}_1(\varphi) := \overline{\varphi},\quad \mathcal{S}_2(\varphi) = \check{\varphi}, \tag{1.2}$$

$$M_\alpha\varphi := \mathrm{e}^{i\alpha}\varphi,\quad T_\tau\varphi := \varphi(\tau+\cdot), \tag{1.3}$$

where $\check{\varphi}$ is defined by $\check{\varphi}(x) = \varphi(-x)$. For convenience, we introduce $\mathcal{S}_3 := M_\pi$, *i.e.* $\mathcal{S}_3(\varphi) = -\varphi$. Notice that the Hamiltonian $\mathcal{H}$ is invariant under $\mathcal{S}_1, \mathcal{S}_2, M_\alpha$ and T_τ.

Denote by $U(t)$ the solution operator of (1.1) for initial data in $L^2(S^1;\mathbb{C})$ (or some Sobolev space $H^N(S^1;\mathbb{C})$) (cf [1]). It is immediate that $U(t)$ commutes with $\mathcal{S}_2, \mathcal{S}_3, M_\alpha$ and T_τ and that

$$U(t)\mathcal{S}_1 = \mathcal{S}_1 U(-t). \tag{1.4}$$

Recall that NLS admits a Lax pair representation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L,A]$$

where $L = L(\varphi)$ is the Zakharov-Shabat operator

$$L(\varphi) := i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix}0 & \varphi\\ \overline{\varphi} & 0\end{pmatrix} \tag{1.5}$$

and A is a (rather complicated) operator given in [2]. We remark that $L(\varphi)$ is unitarily equivalent to the well known AKNS-operator

$$H(\varphi) := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix}-q & p\\ p & q\end{pmatrix} \tag{1.6}$$

where $\varphi = -q + ip$, a fact which will be used throughout the paper.

Denote by $\operatorname{spec}_{\mathrm{per}} L(\varphi)$ the periodic spectrum of $L(\varphi)$ when considered on the interval $[0,2]$ and by $\operatorname{spec}^{\pm}_{\mathrm{Dir}} L(\varphi)$ the Dirichlet spectra of $L(\varphi)$ when considered on the interval $[0,1]$ (*cf.* Definitions (2.5) and (2.6) below). The operator $L(\varphi)$ is selfadjoint when considered with periodic or Dirichlet boundary conditions. Hence both $\operatorname{spec}_{\mathrm{per}} L(\varphi)$ and $\operatorname{spec}_{\mathrm{Dir}} L(\varphi)$ are real.

By elementary considerations one shows that

$$\operatorname{spec}_{\mathrm{per}} L(\overline{\varphi}) = -\operatorname{spec}_{\mathrm{per}} L(\varphi), \quad \operatorname{spec}_{\mathrm{per}} L(\check{\varphi}) = -\operatorname{spec}_{\mathrm{per}} L(\varphi),$$
$$\operatorname{spec}_{\mathrm{per}} L(M_\alpha\varphi) = \operatorname{spec}_{\mathrm{per}} L(\varphi), \quad \operatorname{spec}_{\mathrm{per}} L(T_\tau\varphi) = \operatorname{spec}_{\mathrm{per}} L(\varphi)$$

and expresses $\operatorname{spec}^{+}_{\mathrm{Dir}} L(\mathcal{S}_j\varphi)$ for $j=1,2,3$ in terms of $\operatorname{spec}^{-}_{\mathrm{Dir}} L(\varphi)$.

Recall from [7] (see also [8]) that NLS admits global Birkhoff coordinates. Denote by $\ell^2(\mathbb{Z};\mathbb{R}^2)$ the space of ℓ^2-sequences $(x_j,y_j)_{j\in\mathbb{Z}}$ endowed with the canonical Poisson bracket $\{x_i,x_j\}=0$, $\{y_i,y_j\}=0$ and $\{x_i,y_j\}=\delta_{ij}$.

Theorem 1.1. — *There exists a canonical diffeomorphism* Φ

$$\Phi : \ell^2(\mathbb{Z};\mathbb{R}^2) \longrightarrow L^2(S^1;\mathbb{C})$$

such that

1) Φ *is bianalytic;*

2) *the restriction of* Φ *to the weighted* ℓ^2*-space* $\ell^2_N(\mathbb{Z};\mathbb{R}^2)$ $(N \geq 1)$ *is a diffeomorphism onto the Sobolev space* $H^N(S^1;\mathbb{C})$*;*

3) $(x_j,y_j)_{j\in\mathbb{Z}} = \Phi^{-1}(\phi)$ *are Birkhoff coordinates for NLS and its hierarchy, i.e. any Hamiltonian in the hierarchy is a function of the actions* $I_j := \frac{1}{2}(x_j^2 + y_j^2)$ *only.*

In this article we use the explicit formulas for action and angle variables given in [8] (see also [7]) to obtain

Theorem 1.2. — (i) *The actions are left invariant by* M_α *and* T_τ*, i.e. for any* $k\in\mathbb{Z}$

$$I_k(M_\alpha\varphi) = I_k(\varphi) \quad \text{and} \quad I_k(T_\tau\varphi) = I_k(\varphi)$$

whereas $I_k(\overline{\varphi})$ *and* $I_k(\check{\varphi})$ *can be computed to be* $(j=1,2)$

$$I_k(\mathcal{S}_j\varphi) = I_{-k}(\varphi).$$

(ii) *For* k *with* $I_k \neq 0$

$$\theta_k(M_\alpha\varphi) \equiv \theta_k + \alpha \pmod{2\pi},$$
$$\theta_k(\check{\varphi}) \equiv \theta_{-k}(\varphi) \pmod{2\pi},$$
$$\theta_k(\overline{\varphi}) \equiv -\theta_{-k}(\varphi) \pmod{2\pi}.$$

As a first application of Theorem 1.2 one obtains (*cf.* Proposition 4.1 in Section 4)

COROLLARY 1.1. — *When evaluated at* $I = (I_k)_{k\in\mathbb{Z}}$ *with* $I_k = I_{-k}$ *for all* $k \in \mathbb{Z}$, *the NLS frequencies* $\omega = (\omega_k)_{k\in Z}$, $\omega_k = \partial\mathcal{H}/\partial I_k$, *are symmetric, i.e.* $\omega_k(I) = \omega_{-k}(I)$ *for all* $k \in \mathbb{Z}$.

The main motivation for proving Theorem 1.2 and Corollary 1.1 comes from an application to a KAM type theorem established in [5] (see also [6]).

As a second application, Theorem 1.2 is used to prove that the periodic spectrum is symmetric if and only if the sequence of the gap lengths is symmetric, a conjecture, raised by several experts in the field. More precisely, denote by

$$\operatorname{spec}_{\mathrm{per}} L(\varphi) = \big(\lambda_k^{\pm}(\varphi)\big)_{k\in\mathbb{Z}}$$

the periodic spectrum of $L(\varphi)$ when considered on the inverval $[0,2]$ where the numbers $\lambda_k^{\pm}(\varphi)$ are ordered so that

$$\lambda_k^-(\varphi) \leq \lambda_k^+(\varphi) < \lambda_{k+1}^-(\varphi)$$

and let $\gamma(\varphi) := (\gamma_k(\varphi))_{k\in\mathbb{Z}}$ be the sequence of gap lengths,

$$\gamma_k(\varphi) := \lambda_k^+(\varphi) - \lambda_k^-(\varphi).$$

In Section 4 we prove the following

THEOREM 1.3. — *For* $\varphi \in L^2(S^1;\mathbb{C})$, *the following assertions are equivalent:*

(i) $\lambda_k^{\pm}(\varphi) = -\lambda_{-k}^{\mp}(\varphi)$ *for any* $k \geq 0$;

(ii) $\gamma_k(\varphi) = \gamma_{-k}(\varphi)$ *for any* $k \geq 1$.

2. Symmetries and spectra

2.1. Periodic spectrum. — The periodic spectrum of the Zakharov-Shabat operator $L(\varphi)$ is given by

$$\operatorname{spec}_{\mathrm{per}} L(\varphi) := \big\{\lambda \in \mathbb{C} \mid \exists\, F \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}^2),\ F \not\equiv 0 \ \text{ with } L(\varphi)F = \lambda F \\ \text{and } F(x+2) = F(x),\ \forall x \in \mathbb{R}\big\}.$$

By [4], $\operatorname{spec}_{\mathrm{per}} L(\varphi)$ consists of a sequence of real numbers $(\lambda_k^{\pm}(\varphi))_{k\in\mathbb{Z}}$, which can be ordered in such a way that (for all $k \in \mathbb{Z}$)

$$\lambda_k^-(\varphi) \leq \lambda_k^+(\varphi) < \lambda_{k+1}^-(\varphi) \tag{2.1}$$

and $\lambda_k^{\pm}(\varphi) \sim k\pi$ for $|k|$ large. We have the following

PROPOSITION 2.1. — *Let* $\varphi \in L^2(S^1;\mathbb{C})$. *Then, for any* $k \in \mathbb{Z}$,

(i) $\lambda_k^{\pm}(\mathrm{e}^{i\alpha}\varphi) = \lambda_k^{\pm}(\varphi), \quad \lambda_k^{\pm}(\varphi) = \lambda_k^{\pm}(T_\tau\varphi) \quad (\forall\alpha \in \mathbb{R},\ \tau \in \mathbb{R})$;

(ii) $\lambda_k^{\pm}(\check{\varphi}) = -\lambda_{-k}^{\mp}(\varphi), \quad \lambda_k^{\pm}(\overline{\varphi}) = -\lambda_{-k}^{\mp}(\varphi)$.

Proof. — (i) For $\alpha \in \mathbb{R}$ arbitrary, define $V_\alpha = \left(\begin{smallmatrix} \mathrm{e}^{-i\alpha/2} & 0 \\ 0 & \mathrm{e}^{i\alpha/2} \end{smallmatrix}\right)$. One easily verifies that

$$L(\mathrm{e}^{i\alpha}\varphi) = V_\alpha^{-1} L(\varphi) V_\alpha \tag{2.2}$$

and $L(T_\tau \varphi) = T_\tau L(\varphi) T_{-\tau}$. Thus both, $L(\mathrm{e}^{i\alpha}\varphi)$ and $L(T_\tau\varphi)$, are unitarily equivalent to $L(\varphi)$ and the claimed statement follows. To prove (ii) notice that

$$L(-\check{\varphi}) = -W^{-1} L(\varphi) W \tag{2.3}$$

where W is the unitary operator defined by

$$W\begin{pmatrix} Y \\ Z \end{pmatrix} := \begin{pmatrix} \check{Y} \\ \check{Z} \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} Y \\ Z \end{pmatrix} \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{C}^2).$$

Thus

$$\operatorname{spec}_{\mathrm{per}} L(-\check{\varphi}) = -\operatorname{spec}_{\mathrm{per}} L(\varphi). \tag{2.4}$$

Combining (2.4) and (i) we obtain $\lambda_k^\pm(\check{\varphi}) = -\lambda_{-k}^\mp(\varphi)$ for all $k \in \mathbb{Z}$. Consider $\lambda \in \operatorname{spec}_{\mathrm{per}} L(\varphi)$ and choose $F \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{C}^2)$, satisfying $F(x+2) = F(x)$ for all $x \in \mathbb{R}$ and $L(\varphi)F = \lambda F$. As λ is real, $L(-\overline{\varphi})\overline{F} = -\lambda\overline{F}$ and thus $-\lambda \in \operatorname{spec}_{\mathrm{per}} L(-\overline{\varphi})$. Combined with (i), this leads to $\lambda_k^\pm(\overline{\varphi}) = -\lambda_{-k}^\mp(\varphi)$ for all $k \in \mathbb{Z}$. □

2.2. Dirichlet spectra and divisors. — To study properties of the Dirichlet spectra it is convenient to consider the AKNS operator $H(\varphi)$ instead of $L(\varphi)$. Let

$$F_j(x, \lambda; \varphi) := \begin{pmatrix} Y_j(x, \lambda; \varphi) \\ Z_j(x, \lambda; \varphi) \end{pmatrix}, \quad j = 1, 2,$$

be the fundamental solutions of $H(\varphi)$, *i.e.* the solutions to $HF = \lambda F$ such that

$$F_1(0, \lambda; \varphi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad F_2(0, \lambda; \varphi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For each $x \in \mathbb{R}$ and $\varphi \in L^2(S^1; \mathbb{C})$, $F_1(x, \lambda; \varphi)$ and $F_2(x, \lambda; \varphi)$ are entire functions of λ. The two Dirichlet spectra are defined as follows

$$\operatorname{spec}^+_{\mathrm{Dir}} L(\varphi) = \big\{\lambda \in \mathbb{C} \mid Z_1(1, \lambda; \varphi) = 0\big\}, \tag{2.5}$$

$$\operatorname{spec}^-_{\mathrm{Dir}} L(\varphi) = \big\{\lambda \in \mathbb{C} \mid Y_2(1, \lambda; \varphi) = 0\big\}. \tag{2.6}$$

It is proved in [4] that $\operatorname{spec}^+_{\mathrm{Dir}} L(\varphi)$, resp. $\operatorname{spec}^-_{\mathrm{Dir}} L(\varphi)$, consists of simple, real eigenvalues $(\mu_k(\varphi))_{k\in\mathbb{Z}}$, resp. $(\nu_k(\varphi))_{k\in\mathbb{Z}}$. The numerotation is chosen in such a way that $(\mu_k(\varphi))_{k\in\mathbb{Z}}$ and $(\nu_k(\varphi))_{k\in\mathbb{Z}}$ are strictly increasing satisfying $\mu_k(\varphi) \sim k\pi$ and $\nu_k(\varphi) \sim k\pi$ for $|k|$ large. Further introduce the function $\delta(\lambda; \varphi)$, defined by

$$\delta(\lambda; \varphi) = Z_2(1, \lambda; \varphi) - Y_1(1, \lambda; \varphi). \tag{2.7}$$