Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

NATURAL ENDOMORPHISMS OF QUASI-SHUFFLE HOPF ALGEBRAS

Jean-Christophe Novelli & Frédéric Patras & Jean-Yves Thibon

Tome 141
Fascicule 1

2013

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 107-130

Bull. Soc. math. France
141 (1), 2013, p. 107–130

NATURAL ENDOMORPHISMS OF QUASI-SHUFFLE HOPF ALGEBRAS

by Jean-Christophe Novelli, Frédéric Patras & Jean-Yves Thibon

Abstract. — The Hopf algebra of word-quasi-symmetric functions (**WQSym**), a noncommutative generalization of the Hopf algebra of quasi-symmetric functions, can be endowed with an internal product that has several compatibility properties with the other operations on **WQSym**. This extends constructions familiar and central in the theory of free Lie algebras, noncommutative symmetric functions and their various applications fields, and allows to interpret **WQSym** as a convolution algebra of linear endomorphisms of quasi-shuffle algebras. We then use this interpretation to study the fine structure of quasi-shuffle algebras (MZVs, free Rota-Baxter algebras...). In particular, we compute their Adams operations and prove the existence of generalized Eulerian idempotents, that is, of a canonical left-inverse to the natural surjection map to their indecomposables, allowing for the combinatorial construction of free polynomial generators for these algebras.

Résumé (*Sur les endomorphismes naturels des algèbres de quasi-shuffle*)
L'algèbre de Hopf des fonctions quasi-symétriques sur les mots (**WQSym**), une généralisation non commutative de l'algèbre de Hopf des fonctions quasi-symétriques,

Texte reçu le 5 janvier 2011, révisé le 29 mars 2012, accepté le 13 juillet 2012.

Jean-Christophe Novelli, Institut Gaspard-Monge, Université Paris-Est, 5, boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France • *E-mail :* novelli@univ-mlv.fr
Frédéric Patras, Laboratoire de Mathématiques J.A. Dieudonné,, Université de Nice - Sophia Antipolis,, Parc Valrose, 06108 Nice Cedex 02,, France • *E-mail :* patras@unice.fr
Jean-Yves Thibon, Institut Gaspard-Monge, Université Paris-Est, 5, boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France • *E-mail :* jyt@univ-mlv.fr

2010 Mathematics Subject Classification. — 05E05; 16W20, 05E15.

Key words and phrases. — Quasi-shuffle, Word quasi-symmetric function, convolution, Hopf algebra, surjection, Adams operation, eulerian idempotent, multiple zeta values.

peut être munie d'un produit interne qui a des propriétés remarquables de compatibilité aux autres opérations sur **WQSym**. Cette construction étend des constructions familières et centrales de la théorie des algèbres de Lie libres, des fonctions non commutatives symétriques et de leurs nombreux domaines d'application. Elle permet aussi d'interpréter **WQSym** comme algèbre de convolution des endomorphismes linéaires des algèbres quasi-shuffle. Nous utilisons cette interprétation pour étudier la structure fine des algèbres quasi-shuffle (MZVs, algèbres de Rota-Baxter libres...). En particulier, nous étudions leurs opérations d'Adams et prouvons l'existence d'un inverse à gauche canonique à la surjection naturelle vers les indécomposables ; elle donne lieu à une construction combinatoire de leurs générateurs polynomiaux.

1. Introduction

Quasi-shuffles appeared as early as 1972 in the seminal approach by P. Cartier to Baxter algebras (now most often called Rota-Baxter algebras) [3]. Their study was revived and intensified during the last 10 years, for a variety of reasons. The first one was the study of MZVs (multiple zeta values), for example in the works of Hoffman, Minh, Racinet or Zagier, since quasi-shuffles encode one representation of their products. Another line of study, largely motivated by the recent works of Connes and Kreimer on the structures of quantum field theories, was the revival of the theory of Rota-Baxter algebras initiated by M. Aguiar, K. Ebrahimi-Fard, L. Guo, and others. We refer to [17, 29, 5, 18, 1, 13, 9, 10], also for further bibliographical and historical references on these subjects. Quasi-shuffle algebras are also the free commutative tridendriform algebras [20]. A systematic presentation of Rota-Baxter algebras and of the above relations can be found in the forthcoming book [12].

The present work arose from the project to understand the combinatorial structure of "natural" operations acting on the algebra of MZVs and, more generally on quasi-shuffle algebras. It soon became clear to us that the Hopf algebra of word quasi-symmetric functions (**WQSym**), was the right setting to perform this analysis and that many properties of the classical Lie calculus (incorporated in the theory of free Lie algebras and connected topics) could be translated into this framework.

This article is a first step in that overall direction. It shows that word quasi-symmetric functions act naturally on quasi-shuffle algebras and that some key ingredients of the classical Lie calculus such as Solomon's Eulerian idempotents can be lifted to remarquable elements in **WQSym**. In the process, we show that **WQSym** is the proper analogue of the Hopf algebra **FQSym** of free quasi-symmetric functions (also known as the Malvenuto-Reutenauer Hopf algebra) in the setting of quasi-shuffle algebras. Namely, we prove a Schur-Weyl duality theorem for quasi-shuffle algebras extending naturally the classical one (which states that the linear span of permutations is the commutant of endomorphims of the tensor algebra over a vector space V induced by linear endomorphims of V).

The main ingredient of this theory is that the natural extension of the internal product on the symmetric group algebras to a product on the linear span of surjections between finite sets, which induces a new product on **WQSym**, is a lift in **WQSym** of the composition of linear endomorphisms of quasi-shuffle algebras. This simple observation yields ultimately the correct answer to the problem of studying the formal algebraic structure of quasi-shuffle algebras from the Lie calculus point of view.

2. Word quasi-symmetric functions

In this section, we briefly survey the recent theory of noncommutative quasi-symmetric functions and introduce its fundamental properties and structures. The reader is referred to [15, 6, 16] for details and further information. Let us mention that the theory of word quasi-symmetric functions is very closely related to the ones of Solomon-Tits algebras and twisted descents, the development of which was motivated by the geometry of Coxeter groups, the study of Markov chains on hyperplane arrangements and Joyal's theory of tensor species. We will not consider these application fields here and refer to [35, 2, 31, 28].

Let us first recall that the Hopf algebra of noncommutative symmetric functions [11] over an arbitrary field $\mathbb{K}$ of characteristic zero, denoted here by **Sym**, is defined as the free associative algebra over an infinite sequence $(S_n)_{n\geq 1}$, graded by $\deg S_n = n$, and endowed with the coproduct

$$\Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k} \quad (\text{where } S_0{=}1)\,. \tag{1}$$

It is naturally endowed with an internal product $*$ such that each homogeneous component $\mathbf{Sym}_n$ gets identified with the (opposite) Solomon descent algebra of $\mathfrak{S}_n$, the symmetric group of order n. Some bigger Hopf algebras containing **Sym** in a natural way are also endowed with internal products, whose restriction to **Sym** coincides with $*$. An almost tautological example is **FQSym**, which, being based on permutations, with the group law of $\mathfrak{S}_n$ as internal product, induces naturally the product of the descent algebra [7, 21].

A less trivial example [22] is $\mathbf{WQSym}^*$, the graded dual of **WQSym** (Word Quasi-symmetric functions, the invariants of the quasi-symmetrizing action on words [6]). It can be shown that each homogenous component $\mathbf{WQSym}_n^*$ can be endowed with an internal product (with a very simple combinatorial definition), for which it is anti-isomorphic with the Solomon-Tits algebra, so that it contains $\mathbf{Sym}_n$ as a $*$-subalgebra in a non-trivial way. The internal product of $\mathbf{WQSym}^*$

is itself induced by the one of **PQSym** (parking functions), whose restriction to the Catalan subalgebra **CQSym** again contains **Sym** in a nontrivial way [23].

Let us recall the relevant definitions. We denote by $A = \{a_1 < a_2 < \ldots\}$ an infinite linearly ordered alphabet and A^* the corresponding set of words. The *packed word* $u = \mathrm{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \cdots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the noncommutative polynomial

$$\mathbf{M}_u(A) := \sum_{\mathrm{pack}(w)=u} w\,. \tag{2}$$

For example, restricting A to the first five integers,

$$\begin{aligned} \mathbf{M}_{13132}(A) = {} & 13132 + 14142 + 14143 + 24243 \\ & + 15152 + 15153 + 25253 + 15154 + 25254 + 35354. \end{aligned} \tag{3}$$

As for classical symmetric functions, the nature of the ordered alphabet A chosen to define word quasi-symmetric functions $\mathbf{M}_u(A)$ is largely irrelevant provided it has enough elements. We will therefore often omit the A-dependency and write simply $\mathbf{M}_u$ for $\mathbf{M}_u(A)$, except when we want to emphasize this dependency (and similarly for the other types of generalized symmetric functions we will have to deal with).

Under the abelianization $\chi :\ \mathbb{K}\langle A\rangle \to \mathbb{K}[A]$, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions M_I, where $I = (|u|_a)_{a\in A}$ is the composition (that is, the sequence of integers) associated with the so-called evaluation vector $\mathrm{ev}(u)$ of u ($\mathrm{ev}(u)_i := |u|_{a_i} := |\{j, u_j = a_i\}|$). Recall, for the sake of completeness, that the M_I are defined, for $I = (i_1, ..., i_k)$, by:

$$M_I := \sum_{j_1 < \ldots < j_k} a_{j_1}^{i_1} \ldots a_{j_k}^{i_k}. \tag{4}$$

The polynomials $\mathbf{M}_u$ span a subalgebra **WQSym** of $\mathbb{K}\langle A\rangle$ [14]. This algebra can be understood alternatively as the algebra of invariants for the noncommutative version [6] of Hivert's quasi-symmetrizing action, which is defined in such a way that two words are in the same $\mathfrak{S}(A)$-orbit (where $\mathfrak{S}(A)$ is the group of set automorphisms of A) iff they have the same packed word. We refer to [15] for details on the quasi-symmetrizing action.

As for **Sym**, **WQSym** carries naturally a Hopf algebra structure. Its simplest definition is through the use of two ordered countable alphabets, say $A = \{a_1 < \ldots < a_n < \ldots\}$ and $B := \{b_1 < \ldots < b_n < \ldots\}$. Let us write $A + B$ for the ordinal sum of A and B (so that for arbitrary i, j, we have $a_i < b_j$). The unique associative algebra map μ from $\mathbb{K}\langle A + B\rangle$ to $\mathbb{K}\langle A\rangle \otimes \mathbb{K}\langle B\rangle$ acting