# Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE


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Tome 144
Fascicule 3

2016

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

pages 477-506

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 144, septembre 2016



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*Vente au numéro :* 43 € ($ 64)
*Abonnement* Europe : 178 €, hors Europe : 194 € ($ 291)
Des conditions spéciales sont accordées aux membres de la SMF.

***Secrétariat : Nathalie Christiaën***

*Bulletin de la Société Mathématique de France*
Société Mathématique de France
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revues@smf.ens.fr • http://smf.emath.fr/



ISSN 0037-9484

Directeur de la publication : Stéphane SEURET

*Bull. Soc. math. France*
**144** (3), 2016, p. 477–506

# MONODROMIES AT INFINITY OF NON-TAME POLYNOMIALS

by Kiyoshi Takeuchi & Mihai Tibăr

Abstract. — Polynomials that we usually encounter in mathematics are non-convenient and hence non-tame at infinity. We consider the monodromy at infinity and the monodromies around the bifurcation points of polynomial functions $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ which are non-tame at infinity and might have non-isolated singularities. Our description of their Jordan blocks in terms of the Newton polyhedra and the motivic Milnor fibers relies on two new issues: the non-atypical eigenvalues of the monodromies and the corresponding concentration results for their generalized eigenspaces.

Résumé (*Monodromies à l'infini des polynômes non-modérés*). — Les polynômes qu'on rencontre d'habitude en mathématiques sont généralement non-commodes et donc non-modérés à l'infini. On considère ici la monodromie à l'infini et les monodromies autour les valeurs de bifurcation des fonctions polynômiales $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ qui sont non-modérés à l'infini et peuvent avoir des singularités non-isolées. Notre description de leurs blocs de Jordan en termes des polyèdres de Newton et des fibres de Milnor motiviques s'appuie sur deux nouveaux concepts : les valeurs propres non-atypiques des monodromies et les résultats de concentration pour leurs espaces propres généralisés.

*Texte reçu le 28 novembre 2013, révisé le 8 septembre 2015, accepté le 14 décembre 2015.*

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2010 Mathematics Subject Classification. — 14E18, 14M25, 32C38, 32S35, 32S40.

Key words and phrases. — Atypical values, non-convenient polynomials, monodromy at infinity, Jordan blocks, motivic Milnor fibre, Newton polyhedron, toric compactification.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2016/477/$ 5.00

## 1. Introduction

For a polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$, it is well-known that there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B \tag{1.1}$$

of $f$ is a locally trivial fibration. We denote by $B_f$ the smallest subset $B \subset \mathbb{C}$ satisfying this condition. We call the elements of $B_f$ *bifurcation points of $f$*. For $f(x) = \sum_{v \in \mathbb{Z}^n_+} a_v x^v$ ($a_v \in \mathbb{C}$) we call the convex hull of $\mathrm{supp} f = \{v \in \mathbb{R}^n_+ \mid a_v \neq 0\}$ in $\mathbb{R}^n$ the Newton polytope of $f$ and denote it by $NP(f)$. After Kushnirenko [11], the convex hull $\Gamma_\infty(f) \subset \mathbb{R}^n_+$ of $\{0\} \cup NP(f)$ in $\mathbb{R}^n$ is called the Newton polyhedron at infinity of $f$.

Definition 1.1. — We say that $f$ is *convenient* if $\Gamma_\infty(f)$ intersects the positive part of the $i$-th axis of $\mathbb{R}^n$ for any $1 \leq i \leq n$.

If $f$ is convenient and non-degenerate at infinity (see Definition 2.1), then by a result of Broughton [1] it is *tame at infinity.* In this tame case he proved that one has the concentration

$$H^j(f^{-1}(R); \mathbb{C}) = 0 \quad (j \neq 0, n-1) \tag{1.2}$$

for the generic fiber $f^{-1}(R)$ ($R \gg 0$) of $f$. After this fundamental result many mathematicians studied tame polynomials. However, polynomials that we usually encounter in mathematics are non-convenient and hence non-tame at infinity. According to the fundamental result of Némethi and Zaharia [18], they have a lot of singularities at infinity in general. The study of non-tame polynomials is important for the Jacobian Conjecture since non-tame polynomials are the only interesting objects in the problem. Their study would be useful also in the mirror symmetry, where the Landau-Ginzburg potentials may be non-convenient. Moreover, in what concerns the evaluation of the bifurcation set $B_f$, non-tame polynomials were studied by many mathematicians and with different methods, in particular by Némethi and Zaharia [18], [31] by using Newton polyhedra. The main reason why non-tame polynomials could not be studied precisely before is that one cannot expect to have the concentration (1.2) for them.

In this paper we overcome this difficulty on non-tame polynomials by improving the above-mentioned result of Broughton [1]. Let $C_R = \{x \in \mathbb{C} \mid |x| = R\}$ ($R \gg 0$) be a sufficiently large circle in $\mathbb{C}$ such that $B_f \subset \{x \in \mathbb{C} \mid |x| < R\}$. Then by restricting the locally trivial fibration $\mathbb{C}^n \setminus f^{-1}(B_f) \longrightarrow \mathbb{C} \setminus B_f$ to $C_R$ we obtain a geometric monodromy automorphism $\Phi_f^\infty\colon f^{-1}(R) \xrightarrow{\sim} f^{-1}(R)$ and the linear maps

$$\Phi_j^\infty\colon H^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R); \mathbb{C}) \quad (j = 0, 1, \ldots) \tag{1.3}$$

associated to it, where the orientation of $C_R$ is taken to be counter-clockwise as usual. We call $\Phi_j^{\infty}$'s the (cohomological) *monodromies at infinity of $f$*. In the last few decades many mathematicians studied $\Phi_j^{\infty}$'s from various points of view. In the tame case, Libgober-Sperber [12] obtained a beautiful formula which expresses the semisimple part (i.e., the eigenvalues) of $\Phi_{n-1}^{\infty}$ in terms of the Newton polyhedron at infinity of $f$ (see [13] for its generalizations). Recently in [14] (see also [6]) the first author proved formulae for its nilpotent part, i.e., its Jordan normal form, by using the motivic Milnor fiber at infinity of $f$. However, the methods of [12], [13] and [14] etc. do not apply beyond the tame case by the absense of the concentration (1.2) for non-tame polynomials (see [16] for a partial result). In this paper, even for non-tame polynomials we show that the desired cohomological concentration holds for the generalized eigenspaces of $\Phi_j^{\infty}$ for "good" eigenvalues associated to $f$. Then by avoiding the remaining "bad" eigenvalues, we can successfully generalize the results in [12], [13] and [14] etc. to non-tame polynomials and completely determine the Jordan normal forms of $\Phi_{n-1}^{\infty}$. More precisely, in Definition 2.10 by using the Newton polyhedron at infinity $\Gamma_{\infty}(f)$ we define a finite subset $A_f \subset \mathbb{C}$ of "bad" eigenvalues which we call *atypical engenvalues* of $f$. Then we have the following refinement of the main result of Broughton [1]. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let $H^j(f^{-1}(R);\mathbb{C})_{\lambda} \subset H^j(f^{-1}(R);\mathbb{C})$ be the generalized eigenspace for the eigenvalue $\lambda$ of the monodromy at infinity $\Phi_j^{\infty}$.

THEOREM 1.2. — *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-convenient polynomial such that* $\dim\Gamma_{\infty}(f) = n$. *Assume that $f$ is non-degenerate at infinity. Then for any non-atypical eigenvalue $\lambda \notin A_f$ of $f$ we have the concentration*

$$H^j(f^{-1}(R);\mathbb{C})_{\lambda} \simeq 0 \qquad (j \neq n-1) \tag{1.4}$$

*for the generic fiber $f^{-1}(R) \subset \mathbb{C}^n$ $(R \gg 0)$ of $f$.*

This theorem allows non-isolated singularities of $f$ and also the situation where the fibers may have cohomological perturbation "at infinity". Indeed, some of its atypical fibers $f^{-1}(b)$ $(b \in B_f)$ e.g., $f^{-1}(0)$ have non-isolated singularities in general. In the "tame" case one has only isolated singularities in $\mathbb{C}^n$ and either vanishing cycles at infinity do not occur at all or they occur at isolated points only (in the sense of [25], [29]), and then the concentration of cohomology (1.2) follows.

Theorem 1.2 will be proved by refining the proof of Sabbah's theorem [24, Theorem 13.1] in our situation. More precisely we construct a new compactification $\widetilde{X_{\Sigma}}$ of $\mathbb{C}^n$ and study the "horizontal" divisors at infinity for $f$ in $\widetilde{X_{\Sigma}} \setminus \mathbb{C}^n$ very precisely to prove the concentration. With this main result at hand, by using the results in [14, Section 2] we can prove the generalizations of [12], [13] and [14, Theorems 5.9, 5.14 and 5.16] etc. to non-tame polynomials and