TOPOLOGICAL AND SYMBOLIC DYNAMICS

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PREFACE

Mathematical theory of dynamical systems originated in Newton mechanics. In Newton theory of planetary motions, the sun and the planets are regarded as mass points — particles with mass but no volume. They are mutually attracted by force inversely proportional to their distance and move according to the Newton law of motion $F = ma$.

Newton mechanics is deterministic. The positions and momenta of all particles at present time determine uniquely their positions and momenta at arbitrary later time and can be obtained by solving a system of differential equations. The problem to find this solution, to express the positions and momenta as functions of time, is the famous n-body problem. If $n = 2$, the two bodies move around their common center of gravity and their relative motion satisfies the Kepler laws. For $n \geqslant 3$ no simple solution has been found and, starting with Poincaré [**128**], the problem has been studied with geometrical and topological methods.

The key concept in the geometric approach is that of the phase space or *state space*. The state of the system of n particles consists of all their positions and momenta. Since both the position and momentum of a particle is represented by a three-dimensional vector, we need $6n$ real numbers to specify a state of n particles. The state of the system is therefore regarded as a point in an abstract $6n$-dimensional space. The change of state is conceived as movement in the state space and the curve in the state space which represents this movement is called *trajectory*. There exists a function F which assigns to a state x of the system at time 0 the state $F(x, t)$ at time t. Such a function is called a *flow*.

Topological dynamics studies geometrical and topological properties of trajectories. For example, in the movement of planets there are many periodicities. In planetary system the mass of any planet is significantly smaller than that of the sun, and the dominant force which acts on a planet is the attraction to the sun, while attracting forces to other planets are much smaller. The course of a planet around the sun is therefore very close to the Keplerian elliptic motion. After one orbit, the position and momentum of a planet is very close to its initial state and momentum. If the ratio of orbit times of two planets is close to a rational number p/q, then one planet makes p orbits in nearly the same time as the other planet makes q orbits, and the state of the system goes very closely to the initial state. The same phenomenon can be observed in whole planetary system. A trajectory repeatedly visits any arbitrarily small neighbourhood of the initial state. Moreover, the time gaps between successive visits are nearly constant or at least bounded (Figure 1 top left). Trajectories with this property are called *almost periodic* and systems whose any trajectory is almost periodic are called *minimal* (Birkhoff [**18, 19**]).

A quite different behaviour is exhibited by thermodynamic systems. The state of a thermodynamic system is a vector of macroscopic variables like temperature, volume, pressure or concentrations of constituent components. The state is again conceived as a point in a multidimensional state space. An isolated thermodynamic system tends to equilibrium. This is a particular state which is approached by all trajectories. We say that the equilibrium is an *attracting stationary point* (Figure 1 top right).

In a generalized thermodynamics of open systems studied by Prigogine and his school [**132**], the dynamics might be much more complicated than a simple approach to equilibrium. An open thermodynamic system exchanges with its environment both energy and matter and these flows are either constant or in some way controlled. An open thermodynamic system may approach a *limit cycle* in which the state variables oscillate periodically (Figure 1 bottom left). This is the case of Belousov-Zhabotinsky reaction or its mathematical model, the Brusellator.

An attracting fixed point or a limit cycle are examples of an *attractor*. An attractor is a subset of the state space, such that all trajectories which start in its neighbourhood tend to it. An attractor itself is a dynamical system, a *subsystem* of the original system. A dynamical system might have several attractors each with its own *basin of attraction*. In contrast, minimal systems do not have attractors nor subsystems other than themselves.

In even more complex thermodynamic systems, we observe *chaotic attractors* with irregular dynamics. An example is the Lorenz attractor [**101**] developed as a meteorological model (Figure 1 bottom right). Like minimal systems, chaotic systems have trajectories which repeatedly visit any region of the state space. In contrast to minimal systems, however, the time gaps between successive visits are irregular and not bounded. Moreover, not all trajectories have this property. There are many periodic trajectories which visit only small regions of the state space. Another significant property of chaotic systems is their *sensitivity* to initial conditions. If two trajectories start in close initial states, their distance slowly grows and finally the trajectories separate and become independent.

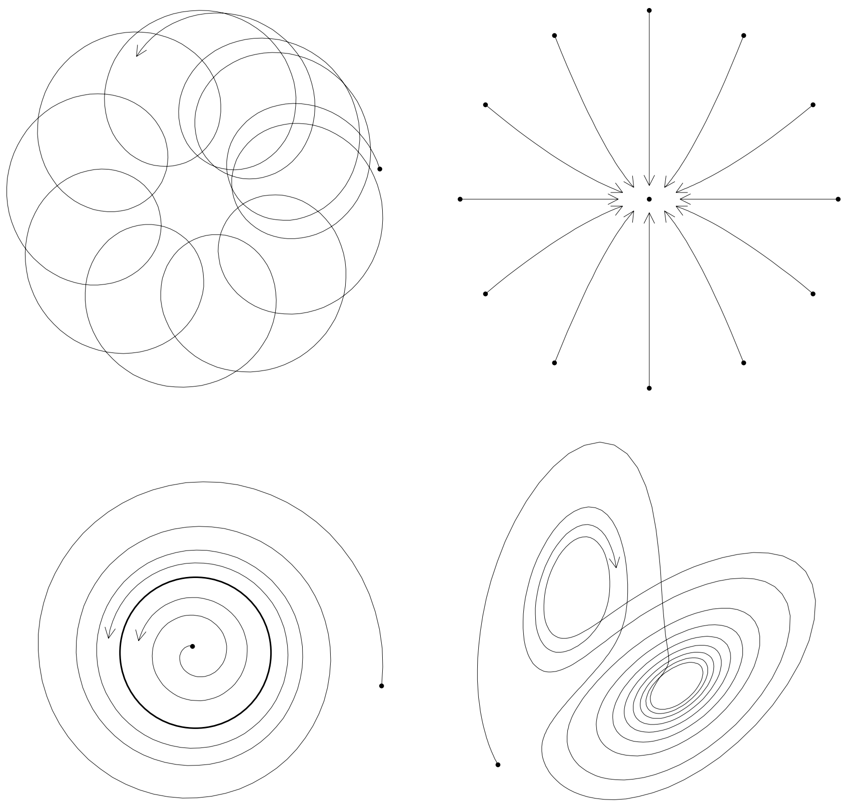

FIGURE 1. Minimal system, attracting state, limit cycle and chaotic system.

Besides classical mechanics and thermodynamics, dynamical systems have been used in many other sciences, for example in populational biology, economics or sociology. Not all these models are formulated as systems of differential equations. There are also dynamical systems with discrete time, where the time variable takes only integer values. This is natural in description of seasonal phenomena, e.g., numbers of migrating animals which come only for a short season in a year. The essential feature of deterministic dynamics (as opposed to the probabilistic dynamics studied in stochastic processes) is that the state at present time determines the state at any future time. Thus there exists a function $F : X \to X$ which assigns to the present state $x \in X$ the state $F(x) \in X$ at time 1. The state at time 2 is then $F(F(x)) = F^2(x)$. Function $F^2 = F \circ F$ is called the second iteration of F and in general, the *n-th iteration* $F^n : X \to X$ is the composition of F with itself n times. A *trajectory* of a discrete time system is a sequence of states $x_n \in X$ given by the recursive formula

$$x_{n+1} = F(x_n), \text{ or } x_n = F^n(x_0).$$

Some systems are deterministic not only with respect to the future but also with respect to the past. From their present state we can determine their state at arbitrary past time. In this case the function F is bijective, and the state at time -1 is $F^{-1}(x)$.

There are many similarities between discrete time and continuous time dynamical systems. The concepts of almost periodicity, minimality, chaoticity or attractors can be studied in both classes by similar methods. However, discrete time systems are conceptually simpler and they can be more easily visualized, especially if the state space is only one-dimensional.

For this reason we start in Chapter 1 with a particular one-dimensional discrete time dynamical system called the logistic or *quadratic system*, which originated in population biology. The system is given by a very simple quadratic equation. Despite the simplicity of its definition, it displays a very wide range of different dynamic behaviours ranging from dynamics dominated by a single attracting periodic orbit to chaotic dynamics. Between these two cases there is the Feigenbaum system characterized by the presence of a minimal subsystem which attracts nearly all orbits. We show how these systems can be understood via symbolic systems, the binary shift in the case of chaotic system and adding machine in the case of Feigenbaum system. Then we treat rotations of the circle as examples of minimal systems and give a nice application of this theory in spiral phyllotaxis.

In Chapter 2 we present topological dynamics based on *dynamical relations* of Akin [**3**]. A dynamical system is defined as a pair (X, F), where X is a compact metric space and $F : X \to X$ is a continuous map. The abstract setting of compact metric spaces leads to an elegant theory. The key assumption of compactness is necessary in order to get interesting theorems. Dynamical relations are natural starting point for the study of minimal, transitive and chain-transitive systems. The concepts of almost periodic, recurrent, nonwandering, and chain-recurrent point also arise very naturally in this setting. Next come the related concepts of equicontinuity and sensitivity, and structural theorems which illustrate their connections to minimality. Minimal equicontinuous systems are studied with the help of spectral theory. We consider the space $\boldsymbol{C}(X, \mathbb{C})$ of complex-valued functions defined on the state space X. A dynamical system induces a linear transformation on $\boldsymbol{C}(X, \mathbb{C})$ and the eigenvalues of this transformation characterize the dynamical system completely.

Then we study attractors and their basins. The theory of attractors relies on the concept of chain relation. An attractor is completely determined by the chain-recurrent points which it contains. The order on the set of chain-recurrent points yields an order on the family of attractors. Finally we study dynamical systems from the point of view of information sources. The amount of information which a dynamical system generates per step is called topological entropy.