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D. CALAQUE & M. GONZALEZ

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Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
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Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
memoires@smf.emath.fr • http://smf.emath.fr/

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ELLIPSITOMIC ASSOCIATORS

Damien Calaque
Martin Gonzalez

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D. Calaque
IMAG, Univ Montpellier, CNRS, Montpellier, France.
E-mail : damien.calaque@umontpellier.fr

M. Gonzalez
Institut de Recherche Technologique SystemX, Palaiseau, France.
E-mail : martin.gonzalez@irt-systemx.fr

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ELLIPSITOMIC ASSOCIATORS

Damien Calaque, Martin Gonzalez

Abstract. – We develop a notion of ellipsitomic associators by means of operad theory. We take this opportunity to review the operadic point-of-view on Drinfeld associators and to provide such an operadic approach for elliptic associators too. We then show that ellipsitomic associators do exist, using the monodromy of the universal ellipsitomic KZB connection, that we introduced in a previous work. We finally relate the KZB ellipsitomic associators to certain Eisenstein series associated with congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, and to twisted elliptic multiple zeta values.

Résumé **(Associateurs ellipsitomiques).** – Nous développons la notion d'associateur ellipsitomique au moyen de la théorie des opérades. Nous saisissons cette opportunité pour revoir le point de vue opéradique sur les associateurs de Drinfeld, et pour fournir également une telle approche opéradique pour les associateurs elliptiques. Nous montrons ensuite que les associateurs ellipsitomiques existent, en utilisant la monodromie de la connexion KZB ellipsitomique universelle, que nous avions introduite dans un travail précédent. Nous relions pour finir les associateurs ellipsitomiques KZB à certaines séries d'Eisenstein associées aux sous-groupes de congruence de $\mathrm{SL}_2(\mathbb{Z})$, et aux valeurs zêta multiples elliptiques tordues.

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INTRODUCTION

The torsor of associators was introduced by Drinfeld [**17**] in the early nineties, in the context of quantum groups and prounipotent Grothendieck-Teichmuller theory. Since then, it has proven to have deep connections with several areas of mathematics (and physics): number theory [**34**], deformation quantization [**22, 33, 39**], Chern-Simons theory and low-dimensional topology [**32**], algebraic topology and the little disks operad [**38**], Lie theory and the Kashiwara-Vergne conjecture [**1, 2**] etc. In this paper we are mostly interested in the operadic and also number theoretic aspects. For instance,

(a) The torsor of associators can be seen as the torsor of isomorphisms between two operads in (prounipotent) groupoids related to the little disks operad, denoted **PaB** and **PaCD** (for *parenthesized braids* and *parenthesized chord diagrams*). These can be understood as the Betti and de Rham fundamental groupoids of an operad of suitably compactified configuration spaces of points in the plane. See Chapter 2 for more details, and accurate references.

(b) It is expected that associators can be seen as generating series for (variations on motivic) multiple zeta values (MZVs), as was observed for the KZ associator [**34**] and the Deligne associator [**11**].

The first example of an associator was produced by Drinfeld as the renormalized holonomy of a universal version of the so-called Knizhnik-Zamolodchikov (KZ) connection [**17**], which is defined on a trivial principal bundle over the configuration space of points in the plane. The defining equations of an associator can be deduced from intuitive geometric reasonings about paths on configuration spaces, and they lead to representations of braid groups.

Enriquez, Etingof and the first author [**12**] introduced a universal version of an elliptic variation on the KZ connection (known as Knizhnik-Zamolodchikov-Bernard, or KZB, connection, as the extension to higher genus is due to Bernard [**6, 5**]). It is a holomorphic connection defined on a non trivial principal bundle over configuration spaces of points on an elliptic curve. They showed that

— The holonomy of the universal KZB connection along fundamental cycles of an ellitpic curve satisfy relations which lead to representations of braid groups on the (2-)torus.

— They also satisfy a modularity property, that is a consequence of the fact that the (universal) KZB connection extends from configuration spaces of points on an elliptic curve to moduli spaces of marked elliptic curves (see also [**35**] for when there are at most 2 marked points).

Enriquez later introduced the notion of an elliptic associator [**19**], and proved that the holonomy of the universal elliptic KZB connection does produce, for every elliptic curve, an example of elliptic associator. The class of elliptic associators that are obtained *via* this procedure are called *KZB associators*. In another work [**21**], Enriquez defined and studied an elliptic version of MZVs; he showed that KZB associators are generating series for elliptic MZVs (eMZVs).

In a recent paper [**13**] we introduced a generalization of the universal elliptic KZB connection: the universal *ellipsitomic* KZB connection. It is defined over twisted configuration spaces, where the twisting is by a finite quotient Γ of the fundamental group of the elliptic curve. When $\Gamma = 1$ is trivial, one recovers the universal elliptic KZB connection.

The aim of the present paper is two-fold.

(a) First we provide an operadic interpretation of elliptic associators. We extend this approach to the ellipsitomic case, use the language of operads to define ellipsitomic associators, and sketch the rudiments of an ellipsitomic Grothendieck-Teichmüller theory.

(b) Then we show that holonomies of the universal ellipsitomic KZB connection along suitable paths produce examples of ellipsitomic associators, and are generating series for elliptic multiple polylogarithms at Γ-torsion points, that are similar to the twisted elliptic MZVs (teMZVs) studied in [**10**] by Broedel-Matthes-Richter-Schlotterer.

Our work fits in a more general program that aims at studying associators for an oriented surface together with a finite group acting on it. We summarize in the following table the contributions to this program that we are aware of:

gen.	group	associators	operadic approach	Universal connection / existence proof	coefficients
0	trivial	[**17**]	[**4, 23**]	rational KZ [**17**] / *ibid.*	MZVs [**34**]
0	$\mathbb{Z}/N\mathbb{Z}$	cyclotomic associators [**18**]	[**14**]	trigonometric KZ [**18**] / *ibid.*	colored MZVs [**18**]
0	fin. $\subset PSU_2(\mathbb{C})$	unknown	unknown	[**36**] / unknown	unknown
1	trivial	elliptic associators [**19**]	this paper (Sec. 3)	elliptic KZB [**12**] / [**19**]	eMZVs [**21**]
1	$\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$	ellipsitomic associators (this paper)	this paper (Sec. 4 & 5)	ellipsitomic KZB [**13**] / this paper (Sec. 6)	this paper (Sec. 7)
> 1	trivial	[**27**]	[**27**]	KZB [**20**] / conj. in [**27**]	maybe [**25**]

Description of the paper

The first chapter is devoted to some recollection on operads and operadic modules, with some emphasis on specific features when the underlying category is the one of groupoids. Chapter 2 also recollects known results, about the operadic approach to (genuine) associators and to various Grothendieck-Teichmüller groups. The main results we state are taken from the recent book **[23]**.

The main goal of Chapter 3 is to provide a similar treatment of elliptic associators, using operadic modules in place of sole operads. We show in particular that (a variant of) the universal elliptic structure $\mathbf{PaB}_{e\ell\ell}$ (resp. its graded/de Rham counterpart $G\mathbf{PaCD}_{e\ell\ell}$) from **[19]** carries the structure of an operadic module in groupoids over the operad in groupoid $\mathbf{PaB}$ (resp. $G\mathbf{PaCD}$). We provide a generators and relations presentation for $\mathbf{PaB}_{e\ell\ell}$ (Theorem 3.3), and deduce from it the following

THEOREM (Theorem 3.15). – *The torsor of elliptic associators from* **[19]** *coincides with the torsor of isomorphisms from (a variant of)* $\mathbf{PaB}_{e\ell\ell}$ *to* $G\mathbf{PaCD}_{e\ell\ell}$ *that are the identity on objects. Similarly, the elliptic Grothendieck-Teichmüller group (resp. its graded version) is isomorphic to the group of automorphisms of* $\mathbf{PaB}_{e\ell\ell}$ *(resp. of* $G\mathbf{PaCD}_{e\ell\ell}$*) that are the identity on objects.*

The fourth chapter introduces a generalization of $\mathbf{PaB}_{e\ell\ell}$, with an additional labeling/twisting by elements of Γ (recall that Γ is the group of deck transformations of a finite cover of the torus by another torus). We give a geometric definition of the operadic module $\mathbf{PaB}^{\Gamma}_{e\ell\ell}$ of parenthesized ellipsitomic braids, and then provide a presentation by generators and relations for it (Theorem 4.5). In the fifth chapter we define an operadic module of ellipsitomic chord diagrams, that mixes features of $\mathbf{PaCD}_{e\ell\ell}$ from Chapter 3, and of the moperad of cyclotomic chord diagrams from **[14]**. This allows us to identify ellipsitomic associators, which we define in purely operadic terms, with series satisfying certain algebraic equations (Theorem 5.9).

Chapter 6 is devoted to the proof of the following

THEOREM (Theorem 6.1). – *The set of ellipsitomic associators over* $\mathbb{C}$ *is non-empty.*

The proof makes crucial use of the ellipsitomic KZB connection, introduced in our previous work **[13]**, and relies on a careful analysis of its monodromy. We actually prove that one can associate an ellipsitomic associator with every element of the upper half-plane (Theorem 6.1). In the last chapter we quickly explore some number theoretic and modular aspects of the coefficients of the "KZB produced" ellipsitomic associators from the previous chapter.

Finally, in an appendix we provide an alternative presentation for $\mathbf{PaB}^{\mathbf{\Gamma}}_{e\ell\ell}$.

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CHAPTER 1

BACKGROUND MATERIAL ON OPERADS AND GROUPOIDS

In this chapter we fix a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ having small colimits. Let us assume for simplicity of exposition that $\otimes$ commutes with these [1].

1.1. $\mathfrak{S}$-modules

An $\mathfrak{S}$-*module* (in $\mathcal{C}$) is a functor $S : \mathbf{Bij} \to \mathcal{C}$, where $\mathbf{Bij}$ denotes the category of finite sets with bijections as morphisms. It can also be defined as a collection $(S(n))_{n \geq 0}$ of objects of $\mathcal{C}$ such that $S(n)$ is endowed with a right action of the symmetric group $\mathfrak{S}_n$ for every n; one has $S(n) := S(\{1, \ldots, n\})$. A *morphism* of $\mathfrak{S}$-modules $\varphi : S \to T$ is a natural transformation. It is determined by the data of a collection $\varphi(n) : S(n) \to T(n)$ of $\mathfrak{S}_n$-equivariant morphisms in $\mathcal{C}$.

The category $\mathfrak{S}$-mod of $\mathfrak{S}$-modules is naturally endowed with a symmetric monoidal product $\otimes$ defined as follows:

$$(S \otimes T)(n) := \coprod_{p+q=n} (S(p) \otimes T(q))^{\mathfrak{S}_n}_{\mathfrak{S}_p \times \mathfrak{S}_q} .$$

Here, if $H \subset G$ is a group inclusion, then $(-)^G_H$ is left adjoint to the restriction functor from the category of objects carrying a G-action to the category of objects carrying an H-action.

The symmetric sequence $\mathbf{1}_\otimes$ defined by

$$\mathbf{1}_\otimes(n) := \begin{cases} \mathbf{1} & \text{if } n = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

is a monoidal unit for $\otimes$.

1. This latter assumption is not necessary (and we will have to get rid of it when considering the monoidal structure given by the direct sum of Lie algebras): if the monoidal product does not commute with colimits, the category of $\mathfrak{S}$-module still has enough structure so that one can define monoids and modules in it. Characterizations in terms of partial compositions remain unchanged. We refer to **[15]** for more details.

There is another (non-symmetric) monoidal product $\circ$ on $\mathfrak{S}$-mod, defined as follows:

$$(S \circ T)(n) := \coprod_{k \geq 0} T(k) \underset{\mathfrak{S}_k}{\otimes} \left(S^{\otimes k}(n)\right).$$

Here, if H is a group and X, Y are objects carrying an H-action, then

$$X \underset{H}{\otimes} Y := \operatorname{coeq}\left(\coprod_{h \in H} X \otimes Y \underset{\mathrm{id} \otimes h}{\overset{h \otimes \mathrm{id}}{\rightrightarrows}} X \otimes Y \right).$$

The symmetric sequence $\mathbf{1}_\circ$ defined by

$$\mathbf{1}_\circ(n) := \begin{cases} \mathbf{1} & \text{if } n = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

is a monoidal unit for $\circ$.

1.2. Operads

An *operad* (in $\mathcal{C}$) is a unital monoid in $(\mathfrak{S}\text{-mod}, \circ, \mathbf{1}_\circ)$. The category of operads in $\mathcal{C}$ will be denoted $\operatorname{Op}\mathcal{C}$.

More explicitly, an operad structure on a $\mathfrak{S}$-module $\mathcal{O}$ is the data:

- of a unit map $e : \mathbf{1} \to \mathcal{O}(1)$;
- for every sets I, J and any element $i \in I$, of a *partial composition*

$$\circ_i : \mathcal{O}(I) \otimes \mathcal{O}(J) \longrightarrow \mathcal{O}\left(J \sqcup I - \{i\}\right)$$

satisfying the following constraints:

- for every sets I, J, K, with elements $i \in I$, $j \in J$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(I) \otimes \mathcal{O}(J) \otimes \mathcal{O}(K) & \xrightarrow{\circ_i \otimes \mathrm{id}} & \mathcal{O}\left(J \sqcup I - \{i\}\right) \otimes \mathcal{O}(K) \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes \circ_j} & & \Big\downarrow{\scriptstyle \circ_j} \\ \mathcal{O}(I) \otimes \mathcal{O}\left(K \sqcup J - \{j\}\right) & \xrightarrow{\circ_i} & \mathcal{O}\left(K \sqcup J \sqcup I - \{i, j\}\right) \end{array}$$

- for every sets I, J_1, J_2, with elements $i_1, i_2 \in I$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(I) \otimes \mathcal{O}(J_1) \otimes \mathcal{O}(J_2) & \xrightarrow{\circ_{i_1} \otimes \mathrm{id}} & \mathcal{O}\left(J_1 \sqcup I - \{i_1\}\right) \otimes \mathcal{O}(J_2) \\ \Big\downarrow{\scriptstyle (\circ_{i_2} \otimes \mathrm{id})(23)} & & \Big\downarrow{\scriptstyle \circ_{i_2}} \\ \mathcal{O}\left(J_2 \sqcup I - \{i_2\}\right) \otimes \mathcal{O}(J_1) & \xrightarrow{\circ_{i_1}} & \mathcal{O}\left(J_2 \sqcup J_1 \sqcup I - \{i_1, i_2\}\right) \end{array}$$

— for every sets I, I', J, $i \in I$, with a bijection $\sigma : I \to I'$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(I) \otimes \mathcal{O}(J) & \xrightarrow{\mathcal{O}(\sigma)} & \mathcal{O}(I') \otimes \mathcal{O}(J) \\ \big\downarrow{\scriptstyle \circ_i} & & \big\downarrow{\scriptstyle \circ_{\sigma(i)}} \\ \mathcal{O}\left(J \sqcup I - \{i\}\right) & \xrightarrow{\mathcal{O}(\mathrm{id} \sqcup \sigma_{|I-\{i\}})} & \mathcal{O}\left(J \sqcup I' - \{\sigma(i)\}\right) \end{array}$$

— for every set I,with $i \in I$, the following diagrams commute:

$$\begin{array}{ccc} \mathbf{1} \otimes \mathcal{O}(I) & \xrightarrow{e \otimes \mathrm{id}} & \mathcal{O}(\{1\}) \otimes \mathcal{O}(I) \\ & {\scriptstyle \simeq} \searrow & \big\downarrow{\scriptstyle \circ_1} \\ & & \mathcal{O}(I) \end{array} \qquad \begin{array}{ccc} \mathcal{O}(I) \otimes \mathbf{1} & \xrightarrow{\mathrm{id} \otimes e} & \mathcal{O}(I) \otimes \mathcal{O}(\{1\}) \\ {\scriptstyle \simeq}\big\downarrow & & \big\downarrow{\scriptstyle \circ_i} \\ \mathcal{O}(I) & \xrightarrow[i \mapsto 1]{\simeq} & \mathcal{O}\left(I \sqcup \{1\} - \{i\}\right). \end{array}$$

EXAMPLE 1.1. – Let X be an object of $\mathcal{C}$. Then we define, for any finite set I, the set $\underline{\mathrm{End}}(X)(I) := \mathrm{Hom}_{\mathcal{C}}(X^{\otimes I}, X)$. Composition of tensor products of maps provide $\underline{\mathrm{End}}(X)$ with the structure of an operad in sets.

Given an operad in sets $\mathcal{O}$, an *$\mathcal{O}$-algebra in $\mathcal{C}$* is an object X of $\mathcal{C}$ together with a morphism of operads $\mathcal{O} \to \underline{\mathrm{End}}(X)$.

1.3. Example of an operad: Stasheff polytopes

To any finite set I we associate the configuration space

$$\mathrm{Conf}(\mathbb{R}, I) = \{\mathbf{x} = (x_i)_{i \in I} \in \mathbb{R}^I | x_i \neq x_j \text{ if } i \neq j\}$$

and its reduced version

$$\mathrm{C}(\mathbb{R}, I) := \mathrm{Conf}(\mathbb{R}, I)/\mathbb{R} \rtimes \mathbb{R}_{>0}.$$

The Axelrod-Singer-Fulton-MacPherson compactification [2] $\overline{\mathrm{C}}(\mathbb{R}, I)$ of $\mathrm{C}(\mathbb{R}, I)$ is a disjoint union of $|I|$-th Stasheff polytopes **[37]**, indexed by $\mathfrak{S}_I$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{R}, I) := \overline{\mathrm{C}}(\mathbb{R}, I) - \mathrm{C}(\mathbb{R}, I)$ is the union, over all partitions $I = J_1 \coprod \cdots \coprod J_k$, of

$$\partial_{J_1, \ldots, J_k} \overline{\mathrm{C}}(\mathbb{R}, I) := \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{R}, J_i) \times \overline{\mathrm{C}}(\mathbb{R}, k).$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{R}, -)$ with the structure of an operad in topological spaces (where the monoidal structure is given by the cartesian product).

2. We are using the differential geometric compactification from **[3]**, which is an analog of the algebro-geometric one from **[24]**.

One can see that $\overline{\mathrm{C}}(\mathbb{R}, I)$ is actually a manifold with corners, and that, considering only zero-dimensional strata of our configuration spaces, we get a suboperad $\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{R}, -)$ that can be shortly described as follows:

— $\mathbf{Pa}(I)$ is the set of pairs (σ, p) with σ is a linear order on I and p a maximal parenthesization of $\underbrace{\bullet \cdots \bullet}_{|I| \text{ times}}$,

— the operad structure is given by substitution.

Notice that $\mathbf{Pa}$ is actually an operad in sets, and that $\mathbf{Pa}$-algebras are nothing else than *magmas*.

1.4. Modules over an operad: Bott-Taubes polytopes

A *module* over an operad $\mathcal{O}$ (in $\mathcal{C}$) is a right $\mathcal{O}$-module in $(\mathfrak{S}\text{-mod}, \circ, \mathbf{1}_\circ)$. Notice that any operad is a module over itself. We let the reader find the very explicit description of a module in terms of partial compositions, as for operads.

To any finite set I we associate the configuration space

$$\mathrm{Conf}(\mathbb{S}^1, I) = \{\mathbf{x} = (x_i)_{i\in I} \in (\mathbb{S}^1)^I | x_i \neq x_j \text{ if } i \neq j\}$$

and its reduced version

$$\mathrm{C}(\mathbb{S}^1, I) := \mathrm{Conf}(\mathbb{S}^1, I)/\mathbb{S}^1.$$

The Axelrod-Singer-Fulton-MacPherson compactification $\overline{\mathrm{C}}(\mathbb{S}^1, I)$ of $\mathrm{C}(\mathbb{S}^1, I)$ is a disjoint union of $|I|$-th Bott-Taubes polytopes [8], indexed by $\mathfrak{S}_I$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{S}^1, I) := \overline{\mathrm{C}}(\mathbb{S}^1, I) - \mathrm{C}(\mathbb{S}^1, I)$ is the union, over all partitions $I = J_1 \coprod \cdots \coprod J_k$, of

$$\partial_{J_1,\ldots,J_k}\overline{\mathrm{C}}(\mathbb{S}^1, I) := \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{R}, J_i) \times \overline{\mathrm{C}}(\mathbb{S}^1, k).$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{S}^1, -)$ with the structure of a module over the operad $\overline{\mathrm{C}}(\mathbb{R}, -)$ in topological spaces.

One can see that $\overline{\mathrm{C}}(\mathbb{S}^1, I)$ is actually a manifold with corners, and that, considering only zero-dimensional strata of our configuration spaces, we get $\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{S}^1, -)$, which is a module over $\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{R}, -)$.

1.5. Convention: pointed versions

Observe that there is an operad Unit defined by

$$\mathrm{Unit}(n) = \begin{cases} \mathbf{1} & \text{if } n = 0, 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

By convention, all our operads $\mathcal{O}$ will be Unit-pointed and reduced, in the sense that they will come equipped with a specific operad morphism $\mathrm{Unit} \to \mathcal{O}$ that is an isomorphism in arity ≤ 1: $\mathcal{O}(n) \simeq \mathbf{1}$ if $n = 0, 1$. Morphisms of operads are required to be compatible with this pointing.

Now, if $\mathcal{P}$ is an $\mathcal{O}$-module, then it naturally becomes a Unit-module as well, by restriction. By convention, all our modules will be pointed as well, in the sense that they will come equipped with a specific Unit-module morphism Unit $\to \mathcal{P}$ that is an isomorphism in arity ≤ 1. Morphisms of modules are also required to be compatible with the pointing.

The main reason for this convention is that we need the following features, that we have in the case of compactified configuration spaces:

— For operads and modules, we want to have "deleting operations" $\mathcal{O}(n) \to \mathcal{O}(n-1)$ that decrease arity.

— For modules, we want to be able to see the operad "inside" them, i.e., we want to have distinguished morphism $\mathcal{O} \to \mathcal{P}$ of $\mathfrak{S}$-modules.

1.6. Group actions

Let G be a $*$-module in group, where $*$ is the terminal operad: the partial composition $\circ_i$ is a group morphism $G(n) \to G(n+m-1)$.

Example 1.2. – Let Γ be a group, we consider the $\mathfrak{S}$-module in groups $\overline{\Gamma} := \{\Gamma^n/\Gamma^{\text{diag}}\}_{n\geq 0}$, where Γ^{diag} denotes the normal closure of the diagonal subgroup in each Γ^n. It is equipped with the following $*$-module structure: the i-th partial composition is given by the partial diagonal morphism

$$\begin{array}{ccc} \Gamma^n/\Gamma & \longrightarrow & \Gamma^{n+m-1}/\Gamma \\ [\gamma_1, \dots, \gamma_n] & \longmapsto & [\gamma_1, \dots, \gamma_{i-1}, \underbrace{\gamma_i, \dots, \gamma_i}_{m \text{ times}}, \gamma_{i+1}, \dots, \gamma_n]. \end{array}$$

Given an operad $\mathcal{O}$ in $\mathcal{C}$, we say that an $\mathcal{O}$-module $\mathcal{P}$ carries a *G-action* if

— for every $n \geq 0$, there is an $\mathfrak{S}_n$-equivariant left action $G(n) \times \mathcal{P}(n) \to \mathcal{P}(n)$,

— for every $m \geq 0$, $n \geq 0$, and $1 \leq i \leq n$, the partial composition

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{O}(m) \longrightarrow \mathcal{P}(n+m-1)$$

is equivariant along the above group morphism $G(n) \to G(n+m-1)$.

A morphism $\mathcal{P} \to \mathcal{Q}$ of $\mathcal{O}$-modules with G-action is said *G-equivariant* if, for every $n \geq 0$, the map $\mathcal{P}(n) \to \mathcal{Q}(n)$ is $G(n)$-equivariant.

Given a group Γ, we say that an $\mathcal{O}$-module $\mathcal{P}$ carries a *diagonally trivial action of* Γ if it carries a $\overline{\Gamma}$-action.

The quotient $G\backslash\mathcal{P}$ of an $\mathcal{O}$-module $\mathcal{P}$ with a G-action is defined as follows:

— For every $n \geq 0$, $\big(G(n)\backslash\mathcal{P}\big)(n) := G(n)\backslash\mathcal{P}(n)$;

— The equivariance of the partial composition $\circ_i$ tels us that it descends to the quotient

$$\big(G(n)\backslash\mathcal{P}(n)\big)\otimes\mathcal{O}(m)\longrightarrow G(n+m-1)\backslash\mathcal{P}(n+m-1).$$

1.7. Semi-direct products and fake pull-backs

Let **Grpd** be the category of groupoids. For a group G, we denote G-**Grpd** the category of groupoids equipped with a G-action. There is a *semi-direct product* functor

$$\begin{array}{ccc} G\text{-}\mathbf{Grpd} & \longrightarrow & \mathbf{Grpd}_{/G} \\ \mathcal{P} & \longmapsto & \mathcal{P}\rtimes G \end{array}$$

where the group G is viewed as a groupoid with a single object, and where $\mathcal{P}\rtimes G$ is defined as follows:

- Objects of $\mathcal{P}\rtimes G$ are just objects of $\mathcal{P}$;
- In addition to the arrows of $\mathcal{P}$, for every $g\in G$, and for every object $\mathbf{p}$ of P, there is an arrow $g\cdot\mathbf{p}\xrightarrow{g}\mathbf{p}$;
- These new arrows multiply together via the group multiplication of G;
- For every morphism f in P, and every $g\in G$, the relation $gfg^{-1}=g\cdot f$ holds.

NOTATION 1.3. – *We warn the reader that we use all along the paper the following rather unusual convention for arrows in a groupoid, and more generally in a category: we often concatenate arrows rather than composing them.*

In other words, $f_1f_2=f_2\circ f_1$.

There is also a functor $\mathcal{G}$ going in the other direction

$$\mathbf{Grpd}_{/G}\longrightarrow G\text{-}\mathbf{Grpd}$$
$$(\mathcal{Q}\xrightarrow{\varphi}G)\longmapsto\mathcal{G}(\varphi),$$

that one can describe as follows:

- The G-set of objects of $\mathcal{G}(\varphi)$ is the free G-set generated by $\mathrm{Ob}(\mathcal{Q})$;
- A morphism $(g,x)\to(h,y)$ in $\mathcal{G}(\varphi)$ is a morphism $x\xrightarrow{f}y$ in $\mathcal{Q}$ such that $g\varphi(f)=h$.

EXAMPLE 1.4. – The groupoid $\mathcal{G}(\mathrm{B}_n\to\mathfrak{S}_n)$ is the colored braid groupoid $\mathbf{CoB}(n)$ from [**23**, §5.2.8].

REMARK 1.5. – Given an object q of $\mathcal{Q}$, $\mathrm{Aut}_{\mathcal{G}(\varphi)}(g,q)$ is the kernel of the morphism $\mathrm{Aut}_{\mathcal{Q}}(q)\to G$ for every $g\in G$.

These constructions still make sense for modules over a given operad $\mathcal{O}$ whenever G is an operadic $*$-module in groups.

Let $\mathcal{P}, \mathcal{Q}$ be two operads (resp. modules) in groupoids. If we are given a morphism $f : \mathrm{Ob}(\mathcal{P}) \to \mathrm{Ob}(\mathcal{Q})$ between the operads (resp. operad modules) of objects of $\mathcal{P}$ and $\mathcal{Q}$, then (following **[23]**) we can define an operad (resp. operad module) $f^\star\mathcal{Q}$ in the following way:

— $\mathrm{Ob}(f^\star\mathcal{Q}) := \mathrm{Ob}(\mathcal{P})$,

— $\mathrm{Hom}_{(f^\star\mathcal{Q})(n)}(p,q) := \mathrm{Hom}_{\mathcal{Q}(n)}(f(p), f(q))$.

In particular, $f^\star\mathcal{Q}$, which we call the *fake pull-back* of $\mathcal{Q}$ along f, inherits the operad structure of $\mathcal{P}$ for its operad of objects and that of $\mathcal{Q}$ for the morphisms.

REMARK 1.6. – Notice that this is not a pull-back in the category of operads in groupoids.

1.8. Prounipotent completion

Let $\mathbf{k}$ be a $\mathbb{Q}$-ring. We denote by $\mathbf{CoAlg_k}$ the symmetric monoidal category of complete filtered topological coassociative cocommutative counital $\mathbf{k}$-coalgebras, where the monoidal product is given by the completed tensor product $\hat{\otimes}_\mathbf{k}$ over $\mathbf{k}$.

Let $\mathbf{Cat}(\mathbf{CoAlg_k})$ be the category of small $\mathbf{CoAlg_k}$-enriched categories. It is symmetric monoidal as well, with monoidal product $\otimes$ defined as follows:

— $\mathrm{Ob}(C \otimes C') := \mathrm{Ob}(C) \times \mathrm{Ob}(C')$.

— $\mathrm{Hom}_{C\otimes C'}\big((c,c'),(d,d')\big) := \mathrm{Hom}_C(c,d)\hat{\otimes}_\mathbf{k}\mathrm{Hom}_{C'}(c',d')$.

All the constructions of the previous section still make sense, at the cost of replacing the group G with its completed group algebra $\widehat{\mathbf{k}G}$ (which is a Hopf algebra) in the semi-direct product construction.

Considering the cartesian symmetric monoidal structure on $\mathbf{Grpd}$, there is a symmetric monoidal functor

$$\begin{aligned} \mathbf{Grpd} &\longrightarrow \mathbf{Cat}(\mathbf{CoAlg_k}) \\ \mathcal{G} &\longmapsto \mathcal{G}(\mathbf{k}), \end{aligned}$$

defined as follows:

— Objects of $\mathcal{P}(\mathbf{k})$ are objects of $\mathcal{P}$.

— For $a, b \in Ob(\mathcal{P})$,

$$\mathrm{Hom}_{\mathcal{P}(\mathbf{k})}(a,b) = \mathbf{k} \cdot \widehat{\mathrm{Hom}_{\mathcal{P}}}(a,b).$$

Here $\mathbf{k} \cdot \mathrm{Hom}_{\mathcal{P}}(a,b)$ is equipped with the unique coalgebra structure such that the elements of $\mathrm{Hom}_{\mathcal{P}}(a,b)$ are grouplike (meaning that they are diagonal for the coproduct and that their counit is 1), and the "$\widehat{\ }$" refers to the completion with respect to the topology defined by the sequence $\big(\mathrm{Hom}_{\mathcal{I}^k}(a,b)\big)_{k\geq 0}$, where $\mathcal{I}^k$ is the category having the same objects as $\mathcal{P}$ and morphisms lying in the k-th power (for the composition of morphisms) of kernels of the counits of $\mathbf{k} \cdot \mathrm{Hom}_{\mathcal{P}}(a,b)$'s.

— For a functor $F : \mathcal{P} \to \mathcal{Q}$, $F(\mathbf{k}) : \mathcal{P}(\mathbf{k}) \to \mathcal{Q}(\mathbf{k})$ is the functor given by F on objects and by $\mathbf{k}$-linearly extending F on morphisms.

Being symmetric monoidal, this functor sends operads in groupoids to operads in $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$.

Example 1.7. – For instance, viewing $\mathbf{Pa}$ as an operad in groupoid (with only identities as morphisms), then $\mathbf{Pa}(\mathbf{k})$ is the operad in $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$ with same objects as $\mathbf{Pa}$, and whose morphisms are

$$\mathrm{Hom}_{\mathbf{Pa}(\mathbf{k})(n)}(a,b) = \begin{cases} \mathbf{k} & \text{if } a = b \\ 0 & \text{otherwise,} \end{cases}$$

with $\mathbf{k}$ being equipped with the coproduct $\Delta(1) = 1 \otimes 1$ and counit $\epsilon(1) = 1$.

The functor we have just defined has a right adjoint

$$G : \mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}}) \longrightarrow \mathbf{Grpd},$$

that we can describe as follows:

— For C in $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$, objects of $G(C)$ are objects of C.

— For $a, b \in Ob(\mathcal{G})$, $\mathrm{Hom}_{G(C)}(a,b)$ is the subset of grouplike elements in $\mathrm{Hom}_C(a,b)$.

Being right adjoint to a symmetric monoidal functor, it is lax symmetric monoidal, and thus it sends operads (resp. modules) to operads (resp. modules).

We thus get a $\mathbf{k}$-*prounipotent completion* functor $\mathcal{G} \mapsto \hat{\mathcal{G}}(\mathbf{k}) := G\big(\mathcal{G}(\mathbf{k})\big)$ for (operads and modules in) groupoids.

Remark 1.8. – Let $\varphi : G \to S$ be a surjective group morphism, and assume that S is finite. One can prove that the prounipotent completion $\hat{\mathcal{G}}(\varphi)(\mathbf{k})$ of the construction from the previous section is isomorphic to $\mathcal{G}(\varphi(\mathbf{k}))$, where $\varphi(\mathbf{k}) : G(\varphi, \mathbf{k}) \to S$ is Hain's relative completion [**28**]. This essentially follows from that, when S is finite, the kernel of the relative completion is the completion of the kernel.

CHAPTER 2

OPERADS ASSOCIATED WITH CONFIGURATION SPACES (ASSOCIATORS)

2.1. Compactified configuration space of the plane

To any finite set I we associate a configuration space

$$\mathrm{Conf}(\mathbb{C}, I) = \{\mathbf{z} = (z_i)_{i\in I} \in \mathbb{C}^I | z_i \neq z_j \text{ if } i \neq j\}.$$

We also consider its reduced version

$$\mathrm{C}(\mathbb{C}, I) := \mathrm{Conf}(\mathbb{C}, I)/\mathbb{C} \rtimes \mathbb{R}_{>0}.$$

We then consider the Axelrod-Singer-Fulton-MacPherson compactification $\overline{\mathrm{C}}(\mathbb{C}, I)$ of $\mathrm{C}(\mathbb{C}, I)$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{C}, I) = \overline{\mathrm{C}}(\mathbb{C}, I) - \mathrm{C}(\mathbb{C}, I)$ is made of the following irreducible components: for any partition $I = J_1 \coprod \cdots \coprod J_k$ there is a component

$$\partial_{J_1,\ldots,J_k}\overline{\mathrm{C}}(\mathbb{C}, I) \cong \overline{\mathrm{C}}(\mathbb{C}, k) \times \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{C}, J_i).$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{C}, -)$ with the structure of an operad in topological spaces. One can picture the partial operadic composition morphisms as follows: