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NEW RESULTS ON THE CLASSICAL PROBLEM OF PLATEAU ON THE EXISTENCE OF MANY SOLUTIONS

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§ 1 THE PROBLEM OF PLATEAU

The notion of a classical minimal surface is not exactly defined. Generally one understands a classical minimal surface to be a two-dimensional surface of mean curvature-zero in Euclidean N-space. These "classical" surfaces need not to be embedded or immersed. However there is only one type of singularities admitted, the so called "branch points". This notion excludes certain singularities, where different pieces of minimal surfaces build up a system of surfaces intersecting "minimally" at angles of 120^0, their edges possibly meeting at angles of 109^0 ,(as discussed and classified in [47]).

One reason for the choice of this class of surfaces is its link to the theory of analytic functions of one complex variable. Namely, it is easy to show that a minimal surface $F \subset \mathbb{R}^N$ as above allows a conformal parametrization, i.e. for such F there exists a Riemann surface R (or possibly a subset $\phi \subset R$) with a fixed conformal structure and a conformal parametrization $f : \phi \to F \subset \mathbb{R}^N$.
The equation "mean curvature $\equiv 0$ in all regular points of F" implies that "$f : \phi \to \mathbb{R}^N$ is harmonic".
If $N = 2$ and f is harmonic and conformal, then f is complex analytic. Therefore, the existence theorems for minimal surfaces can be understood as a generalization of the Riemann mapping theorem. Many conjectures about minimal surfaces (on boundary behavior, on singularities, on their Jacobi fields) have arisen from the examples in the case $N = 2$. The recent work of A. Fisher and A.J. Tromba on conformal structures indicates that the methods of minimal surfaces theory will shed a new light on the classical Teichmüller theory.

The second reason for the above choice of the definition of a minimal surface are the existing existence theorems. They have their origin in the fact that the equation "mean curvature $\equiv 0$" is the Euler-Lagrange equation for the area function on the space of 2-surfaces with fixed boundary. Therefore, one can construct minimal surfaces with a minimizing procedure. Even if today there exist more general existence theorems (due to Reiffenberg, de Giorgi, Federer, Fleming, Almgren) the subsequent approach is the one where the topological type can be prescribed in advance. We refer to [3], too, for the limitations of this approach.

Theorem 1.1 (J. Douglas [12]): If $\Gamma \subset \mathbb{R}^N$ is a Jordan curve, then Γ bounds a classical minimal surface of the type of the disc, i.e. there exists a continuous parametrization $g : S^1 \to \mathbb{R}^N$ such that $g(S^1) = \Gamma$ and that the harmonic extension $x : \bar{D} \to \mathbb{R}^n$ from g to the unit disc D in $\mathbb{R}^2$ is harmonic and conformal on D and continuous on $\bar{D}$, i.e. $x(\bar{D}) \subset \mathbb{R}^N$ is a classical minimal surface.

There exists a more general theorem (J. Douglas [13]) proving the existence of minimal surfaces of higher connectivity k ($k \geq 1$) and of higher genus g, when the boundary set Γ consists of $(k+1)$ Jordan curves and the parameter domain is of genus $g \geq 1$. Such an existence theorem makes assumptions about Γ so that the infimum of the area on all surfaces bounded by Γ and of genus g is smaller then the infimum on all surfaces of a genus bounded by $g_1 < g$. (See [10], [34], [45]).

Theorem 1.2: A major achievement was the proof that - exactly as in the case of linear elliptic systems - the solutions of the Plateau problem are regular up to the boundary, i.e. the surface is smooth up to the boundary, if the boundary is smooth (H^k, $C^{k+\alpha}$, C^∞, C^ω) (H. Lewy, St. Hildebrandt, J.C.C. Nitsche, R. Hardt and L. Simon). [25, 22, 41, 19].

§ 2 BRANCH POINTS OF CLASSICAL SURFACES

The notion of a classical minimal surface ist not completely satisfactory from the point of view of differential geometry. So, a lot of work went into understanding where branch points are possible for solutions of Plateau's problem under various circumstances. A major success was a theorem of Osserman [42] , with later improvements due to H.W. Alt, R. Gulliver,and Gulliver and L.D. Leslie [1, 16, 17]. We summarize

Theorem 2.1: Let $\Gamma \subset \mathbb{R}^3$ be a Jordan curve and $F = x(D)$ be one of the solutions of the classical Plateau problem, i.e. x minimizes Dirichlet's integral $\int_D (x_u^2 + x_v^2)\, du\, dv$ among all mappings in $\{H^1(D, \mathbb{R}^3) \cap C^0(\bar{D}, \mathbb{R}^3);\ x_{|\partial D}$ a parametrization of $\Gamma\}$. Then x has no interior branch points.
If Γ is a real analytic curve, then x has no branch points on the boundary either, i.e. $F = x(\bar{D})$ is a real analstic immersion of the closed disc.

The idea of the proof is easy to understand. In soap film experiments one never can observe branch points. When looking at a branched surface F with boundary Γ and with one branch point P of order $m \geq 1$ on F, then in the neighborhood of P the surface F looks locally "like" a(m+1)-fold cover of the tangent plane to F through P (which does exist). If looking for an absolutely area minimizing surface with boundary Γ, then this (m+1)-fold cover obviously is not an economic way of using the area, and with some "cutting and pasting" one can decrease the area of the surface. The question is only whether one gets again a surface of the type of the disc. These problems got resolved in the proof of theorem 2.1.

Surprisingly the theorem 2.1 depends heavily on the fact that the surface F is situated in $\mathbb{R}^3$, i.e. has codimension 1; it is wrong in $\mathbb{R}^4$. Namely, H. Federer [14] ovserved that a piece of a complex curve L in $\mathbb{C}^2$(or in $\mathbb{C}^n$) is absolutely

area minimizing when the boundary ∂L is fixed, even if the surface L has branch points. Namely:
An integral current (of even dimension) in $\mathbb{C}^n$ (or in a Kähler manifold), which has a complex tangent space almost everywhere is a minimal current.
Osserman's theorem together with 1.2 gives a solution of the disc type for any Plateau problem in $\mathbb{R}^3$ (one boundary curve) which is immersed. If the boundary curve is knotted, there is no hope for the disc type solution to be embedded. But when giving up the condition of disc type there should be a better answer.It was given by R. Hardt and L. Simon [19].

Theorem 2.2: There exists an a priori bound $b(\Gamma)$ for the genus g depending only on the geometry of a C^2-curve Γ in $\mathbb{R}^3$, such that any such Γ bounds an embedded minimal surface (i.e. a minimal submanifold) of genus $g \leq b(\Gamma)$ which is absolutely area minimizing.

Theorem 2.2 is part of a much broader approach to the boundary regularity of minimal surfaces of codimension 1 in $\mathbb{R}^N$ where classical methods of minimal surface theory and the methods of geometric measure theory meet. That such a bound on the genus is not at all trivial follows from an example of W. Fleming [15], which describes a Jordan curve Γ in $\mathbb{R}^3$-rectifiable but not smooth - such that the problem of least area has no solution with a finite topological type. The estimate of Hardt and Simon is not helpful for deciding whether a specific curve Γ bounds an embedded (absolutely area minimizing) disc. But we now know a large class of curves which bound a minimally embedded disc.

Definition: A smooth Jordan curve Γ in $\mathbb{R}^3$ is called extreme, if Γ is situated on the boundary of a convex body (or more generally in a surface with everywhere non negative mean curvature).

Theorem 2.3: (Meeks-Yau, Almgren-Simon, Tromba-Tomi [2, 28, 51]):
Any extreme curve bounds at least one minimally embedded disc (which is absolutely area minimizing).

There are three very different proofs, the proof [51] not showing that the solution actually is an absolute minimum for the area.
The proof of Meeks and Yau is part of a general study of 3-manifolds, depending on Dehn's lemma and the tower construction of topology. The proof of Tomi and Tromba gives a weaker result, but is very easy. They construct the Hilbert manifold of "all" disc type immersions, use a homotopy argument in it and a closedness property of minimal embeddings due to Gulliver and Spruck [18, 52].
Clearly the class of branched surfaces is much too small to cover all singularities which are met with in soap film experiments. We only refer to the important work of J. Taylor [47, 48], and to recent work of F. Morgan [33].

§ 3 UNIQUENESS THEOREMS

Generally it is easier to prove existence (just by constructing a solution) than to show its uniqueness (you would have to look for solutions anywhere in the function space). There are two classical uniqueness theorems.

Theorem 3.1 (T. Radó): If the boundary curve $\Gamma \subset \mathbb{R}^3$ has a convex projection then the solution of Plateau's problem is unique (and a graph).

The proof depends on the maximum principle. (See e.g. [41]).

Theorem 3.2 (J.C.C. Nitsche [39]): If the boundary curve $\Gamma \subset \mathbb{R}^3$ is real analytic and the total curvature K of Γ satisfies $K \leq 4\pi$, then Γ bounds a unique immersed disc. (Probably there are no solutions of higher genus.)