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CRYSTALLINE COHOMOLOGY OF ALGEBRAIC STACKS AND HYODO-KATO COHOMOLOGY

Martin C. Olsson

Abstract. — In this text we study using stack-theoretic techniques the crystalline structure on the de Rham cohomology of a proper smooth scheme over a p-adic field and applications to p-adic Hodge theory. We develop a general theory of crystalline cohomology and de Rham-Witt complexes for algebraic stacks, and apply it to the construction and study of the (φ, N, G)-structure on de Rham cohomology. Using the stack-theoretic point of view instead of log geometry, we develop the ingredients needed to prove the $C_{\rm st}$-conjecture using the method of Fontaine, Messing, Hyodo, Kato, and Tsuji, except for the key computation of p-adic vanishing cycles. We also generalize the construction of the monodromy operator to schemes with more general types of reduction than semistable, and prove new results about tameness of the action of Galois on cohomology.

***Résumé* (Cohomologie cristalline des champs algébriques et isomorphisme de Hyodo-Kato)**

Dans ce texte, nous étudions, par des techniques « champêtres », la structure cristalline sur la cohomologie de de Rham d'un schéma propre et lisse sur un corps p-adique et ses applications à la théorie de Hodge p-adique. Nous développons une théorie générale de la cohomologie cristalline et des complexes de de Rham-Witt associés aux champs algébriques, et l'appliquons à la construction et à l'étude de la (φ, N, G)-structure sur la cohomologie de de Rham. Nous plaçant du point de vue des champs plutôt que de la géométrie logarithmique, nous développons les ingrédients nécessaires à la démonstration de la conjecture $C_{\rm st}$ suivant la voie de Fontaine, Messing, Hyodo, Kato et Tsuji (en laissant de côté le calcul-clé des cycles évanescents p-adiques). Nous généralisons aussi la construction des opérateurs de monodromie aux schémas au-delà du cas semi-stable, et obtenons de nouveaux résultats sur le caractère modéré de l'action galoisienne sur la cohomologie.

CONTENTS

INTRODUCTION

This text grew out of an attempt to understand the theory of log crystalline cohomology and its application to the C_{st} conjecture of Fontaine and Jannsen using the stack-theoretic techniques introduced in [**62**].

Before explaining the contents of the paper let us briefly review the statements of these conjectures, now proven in different ways by Tsuji [**73**], Faltings [**20, 21**], and Niziol [**53, 54**].

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with ring of integers V and perfect residue field k. Let $K \hookrightarrow \overline{K}$ be an algebraic closure of K, let $K_0 \subset K$ be the field of fractions of the ring of Witt-vectors of the residue field of V, and let $K_0^{ur} \subset \overline{K}$ denote the maximal unramified extension of K_0 in $\overline{K}$. There is a canonical automorphism $\sigma : K_0^{ur} \to K_0^{ur}$ induced by the Frobenius on the residue fields. Let G denote the Galois group $\mathrm{Gal}(\overline{K}/K)$. The group G acts by restriction also on K_0^{ur}.

Let X/K be a smooth proper scheme. Associated to X are the de Rham cohomology groups $H^*_{\mathrm{dR}}(X/K)$ and the p-adic étale cohomology groups $H^*(\overline{X}, \mathbb{Q}_p)$, where $\overline{X}$ denotes the base change of X to $\overline{K}$. The space $H^*_{\mathrm{dR}}(X/K)$ comes equipped with the Hodge filtration Fil_H, and the space $H^*(\overline{X}, \mathbb{Q}_p)$ has a continuous action of the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$. The conjectures of Fontaine concern the relationship between $H^*_{\mathrm{dR}}(X/K)$ and $H^*(\overline{X}, \mathbb{Q}_p)$.

One of the key ingredients in the C_{dR}-conjecture relating these two cohomology theories is the construction of a so-called (φ, N, G)-module structure on $H^*_{\mathrm{dR}}(X/K)$ in the following sense.

Definition 0.1.1. — A (φ, N, G)-*module* is a collection of data (D, φ, N) as follows:

(0.1.1.1) A finite-dimensional K_0^{ur}-vector space D with a σ-linear automorphism φ.

(0.1.1.2) A K_0^{ur}-linear nilpotent endomorphism N of D such that $N\varphi = p\varphi N$.

(0.1.1.3) A continuous semilinear (with respect to the natural action of G on K_0^{ur}) action of G on D such that for all $g \in G$, $\varphi \circ g = g \circ \varphi$ and $N \circ g = g \circ N$.

A *filtered* (φ, N, G)*-module* is a collection of data $(D, \varphi, N, \mathrm{Fil})$, where (D, φ, N) is a (φ, N, G)-module and in addition there is the following structure.

(0.1.1.4) A decreasing, separated, and exhaustive filtration Fil on $D_{\overline{K}} := D \otimes_{K_0^{ur}} \overline{K}$ stable under the diagonal action of G.

The category of filtered (φ, N, G)-modules is denoted $\underline{MF}_{\overline{K}/K}(\varphi, N)$.

If T is a finite dimensional K-vector space, then a *(filtered)* (φ, N, G)*-module structure on* T is a (filtered) (φ, N, G)-module (D, φ, N) together with isomorphisms $\rho_\pi : T \simeq (D \otimes_{K_0^{ur}} \overline{K})^G$ for each choice of uniformizer π in K such that if $\pi' = u\pi$ then

(0.1.1.5)
$$\rho_\pi = \rho_{\pi'} \exp(\log(u)N),$$

where log denotes the p-adic logarithm.

It follows from the proof of the C_{st}-conjecture that the de Rham cohomology $H^*_{\mathrm{dR}}(X/K)$ has a natural filtered (φ, N, G)-structure such that the filtration on $H^*_{\mathrm{dR}}(X/K)$ is the Hodge filtration. More precisely, consider the ring B_{st} of Fontaine [**23**, §3]. Let $\mathrm{Rep}(G)$ denote the category of finite dimensional $\mathbb{Q}_p$-vector spaces with continuous action of the Galois group G. To any Galois representation $V \in \mathrm{Rep}(G)$ one can associate a (φ, N, G)-module as follows. The ring B_{st} comes equipped with a semi-linear Frobenius endomorphism φ, an operator N, and an action of G satisfying certain compatibilities. Furthermore, the choice of a uniformizer $\pi \in K$ defines an inclusion $B_{\mathrm{st}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}$, where B_{dR} is as in [**23**, §1]. In particular, $B_{\mathrm{st}} \otimes_{K_0} K$ inherits a filtration from B_{dR}. For any finite extension $K \subset L \subset \overline{K}$ let $G_L \subset G$ denote the subgroup $\mathrm{Gal}(\overline{K}/L)$. Since $B_{\mathrm{st}}^{G_L} = L_0$ (the ring of Witt vectors of the residue field of L), we can define a K_0^{ur}-space

(0.1.1.6)
$$D_{\mathrm{pst}}(V) := \varinjlim_{K \subset L \subset \overline{K}} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_L}.$$

The operators φ and N on B_{st} induce a (φ, N, G)-module structure on $D_{\mathrm{pst}}(V)$, and the inclusion $D_{\mathrm{pst}}(V) \otimes_{K_0^{ur}} \overline{K} \subset B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ obtained by passing to the limit from the inclusions $B_{\mathrm{st}} \otimes_{L_0} L \subset B_{\mathrm{dR}}$ induces a filtration on $D_{\mathrm{pst}}(V) \otimes_{K_0^{ur}} \overline{K}$. We therefore obtain a functor

(0.1.1.7)
$$D_{\mathrm{pst}} : \mathrm{Rep}(G) \longrightarrow \underline{MF}_{\overline{K}/K}(\varphi, N).$$

There is also a functor

(0.1.1.8)
$$V_{\mathrm{pst}} : \underline{MF}_{\overline{K}/K}(\varphi, N) \longrightarrow \mathrm{Rep}(G)$$

sending (D, φ, N) to the G-representation

(0.1.1.9)
$$V_{\mathrm{pst}}(D, \varphi, N) := \{v \in B_{\mathrm{st}} \otimes_{K_0^{ur}} D | Nv = 0, \varphi(v) = v, \text{ and } v \otimes 1 \in \mathrm{Fil}^0(D \otimes_{K_0^{ur}} \overline{K})\}.$$

For $(D, \varphi, N) \in \underline{MF}_{\overline{K}/K}(\varphi, N)$ one can also define $(D \otimes_{K_0^{ur}} \overline{K})^G$. This is a filtered K-vector space.

For $V \in \mathrm{Rep}(G)$, there is a natural map

$$\alpha : B_{\mathrm{st}} \otimes_{K_0^{ur}} D_{\mathrm{pst}}(V) \longrightarrow B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V. \tag{0.1.1.10}$$

The representation V is called *potentially semistable* if this map α is an isomorphism [**24**, 5.6.1]. By [**24**, 5.6.7] the functor D_{pst} is fully faithful when restricted to the subcategory $\mathrm{Rep}^{\mathrm{pst}}(G) \subset \mathrm{Rep}(G)$ of potentially semistable representations, and if $\underline{MF}^{\mathrm{adm}}_{\overline{K}/K}(\varphi, N) \subset \underline{MF}_{\overline{K}/K}(\varphi, N)$ denotes its essential image, then a quasi-inverse

$$\underline{MF}^{\mathrm{adm}}_{\overline{K}/K}(\varphi, N) \longrightarrow \mathrm{Rep}^{\mathrm{pst}}(G) \tag{0.1.1.11}$$

is provided by V_{pst}.

We also consider the subcategory $\mathrm{Rep}^{\mathrm{st}}(G) \subset \mathrm{Rep}^{\mathrm{pst}}(G)$ consisting of representations V for which the natural map

$$B_{\mathrm{st}} \otimes_{K_0} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G \longrightarrow B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \tag{0.1.1.12}$$

is an isomorphism. Such representations are called *semistable* and the essential image of $\mathrm{Rep}^{\mathrm{st}}(G)$ in $\underline{MF}^{\mathrm{adm}}(\varphi, N)$ is precisely those (φ, N, G)-modules (D, φ, N) for which the action of G on D is trivial (that is, the natural map $K_0^{ur} \otimes_{K_0} D^G \to D$ is an isomorphism). Let D_0 denote the space D^G. The operators φ and N induce operators, denoted by the same letters, on D_0 and the filtration descends to $D_0 \otimes_{K_0} K$.

We can now state the C_{st}-conjecture as follows:

Theorem 0.1.2 **([73, 20, 21, 53])**. — *Let X/K be a smooth proper scheme of dimension d with semistable reduction, and let m be an integer. Then the Galois representation $V = H^m(\overline{X}, \mathbb{Q}_p)$ is semistable and for any choice of uniformizer in K there is an isomorphism $K \otimes_{K_0} D_0 \simeq H^m_{\mathrm{dR}}(X/K)$ compatible with the filtrations (where $H^m_{\mathrm{dR}}(X/K)$ is filtered by the Hodge filtration).*

Using de Jong's alterations theorem [**37**] one can deduce from this the following so-called C_{pst}-conjecture:

Theorem 0.1.3 **([74])**. — *Let X/K be a smooth proper scheme and m an integer. Then the Galois representation $V = H^m(\overline{X}, \mathbb{Q}_p)$ is potentially semistable, and for any choice of uniformizer in K there is an isomorphism $(\overline{K} \otimes_{K_0^{ur}} D_{\mathrm{pst}}(V))^G \simeq H^m_{\mathrm{dR}}(X/K)$ compatible with the filtrations.*

Since one can recover the Galois representation $H^m(\overline{X}, \mathbb{Q}_p)$ from $D_{\mathrm{pst}}(V)$, it is of great interest to understand in more detail the (φ, N, G)-module $D_{\mathrm{pst}}(V)$. In the case of semistable reduction, the module $D_{\mathrm{pst}}(V)$ can be constructed using the theory of log geometry and log crystalline cohomology developed by Fontaine, Illusie, and Kato [**40**]. Unfortunately, the construction in general is based on an abstract "independence of model argument" and de Jong's alterations and so does not directly yield information about the Galois representation.

The starting point for this work is the paper [**62**] which gives a dictionary between logarithmic geometry and "ordinary" geometry of schemes over certain algebraic stacks. This suggests the possibility of giving a construction of the (φ, N, G)-structure