Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

DOUBLE EPW-SEXTICS WITH ACTIONS OF $\mathcal{A}_7$ AND IRRATIONAL GM THREEFOLDS

Simone Billi & Tomasz Wawak

Tome 152
Fascicule 4

2024

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 857-868

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 4, tome 152, décembre 2024

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 160 € ($ 240),
avec supplément papier : Europe 244 €, hors Europe 330 € ($ 421)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
152 (4), 2024, p. 857-868

DOUBLE EPW-SEXTICS WITH ACTIONS OF $\mathcal{A}_7$ AND IRRATIONAL GM THREEFOLDS

by Simone Billi & Tomasz Wawak

Abstract. — We construct two examples of projective hyper-Kähler fourfolds of $K3^{[2]}$-type with an action of the alternating group $\mathcal{A}_7$. They are realized as double EPW-sextics, and this allows us to construct two explicit families of irrational Gushel–Mukai threefolds.

Résumé (*Double EPW-sextiques avec des actions de $\mathcal{A}_7$ et variétés GM irrationnelles de dimension troi*). — Nous construisons deux exemples de variétés hyper-Kählériennes de type $K3^{[2]}$ projectifs avec une action du groupe alterné $\mathcal{A}_7$. Ils sont réalisés comme des doubles EPW-sextiques, ce qui nous permet de construire deux familles explicites de variétés de Gushel-Mukai tridimensionnelles qui sont irrationnels.

Texte reçu le 9 mars 2023, modifié le 25 juin 2024, accepté le 9 juillet 2024.

Simone Billi, University of Genova, Via dodecaneso, 35, 16146 Genova, Italy • *E-mail :* simone.billi@edu.unige.it
Tomasz Wawak, Jagiellonian University, Ulica Łojasiewicza 6, 30-348 Kraków, Poland • *E-mail :* tomasz.wawak@doctoral.uj.edu.pl

Mathematical subject classification (2010). — 14J42, 14J45, 14J50.

Key words and phrases. — Hyper-Kähler manifolds, automorphisms, Gushel–Mukai varieties, irrational.

During the writing of this paper, the second author was supported by the project Narodowe Centrum Nauki 2018/30/E/ST1/00530 and the Priority Research Area SciMat under the program Excellence Initiative – Research University at the Jagiellonian University in Kraków.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2024/857/$ 5.00

doi:10.24033/bsmf.1524sbtw

Introduction

We construct two explicit examples of hyper-Kähler fourfolds of $K3^{[2]}$-type with symplectic actions of the alternating group $\mathcal{A}_7$. According to the classification in [14], this is one of the maximal finite groups that can act on a manifold of this type via symplectic automorphisms. We describe the manifolds as double EPW-sextics. For each of them, the group of symplectic automorphisms $\mathcal{A}_7$ fixes their canonical ample divisor.

Consider a 6-dimensional complex vector space V_6. There is a linear representation of $\mathcal{A}_7$ on $\bigwedge^3 V_6$ that decomposes into the direct sum of two Lagrangian subrepresentations A_1, A_2; we set $\mathbb{A} = A_1$ or A_2. There is a canonical way to associate a double EPW-sextic to a general Lagrangian subspace of $\bigwedge^3 V_6$, as explained in Section 1.2. Our main result can be formulated as follows.

Theorem 0.1. — *The double EPW-sextic associated to the Lagrangian subspace $\mathbb{A}$ is a smooth polarized hyper-Kähler fourfold $(Y_{\mathbb{A}}, H_{\mathbb{A}})$. Moreover,*

$$\mathrm{Aut}_{H_{\mathbb{A}}}(Y_{\mathbb{A}}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathcal{A}_7,$$

where $\mathcal{A}_7$ corresponds to the subgroup of symplectic automorphisms. Furthermore, (Y_{A_1}, H_{A_1}) and (Y_{A_2}, H_{A_2}) are non-isomorphic as polarized manifolds.

We can associate a family of GM varieties to such Lagrangian spaces as explained in Section 1.3. As an application, we show that for $\mathbb{A}$ as above, we have the following result.

Corollary 0.2. — *Any smooth GM threefold associated to $\mathbb{A}$ is irrational.*

The only other known explicit family of irrational GM threefold was constructed in [8], and our result provides other two such families.

This article was primarily inspired by [8], from where many ideas have been taken and adjusted to our case. In fact, the search for symmetric hyper-Kähler manifolds has been of interest in recent years. One can find attempts at classification and explicit constructions in [2, 1] for K3 surfaces, in [14, 20, 21, 9] for hyper-Kähler fourfolds of $K3^{[2]}$-type. Our results should be seen as a continuation of these efforts.

The structure of the paper is the following. In the first section, we recall known facts about hyper-Kähler manifolds and the lattice structure on their second integral cohomology group, EPW-sextics, and GM varieties. In the second section, we outline the construction of our examples and in the third, we prove the irrationality of the associated GM threefolds. In the Appendix, we discuss the computation that we performed with `Macaulay2` [13].

1. Preliminaries

This section is merely a collection of known facts about hyper-Kähler manifolds and associated lattices, double EPW-sextics, and GM varieties.

1.1. Hyper-Kähler manifolds and their lattices. — A hyper-Kähler manifold is a compact, simply connected complex Kähler manifold such that $\mathrm{H}^{2,0}(X) \cong \mathrm{H}^0(X, \bigwedge^2 \Omega_X) \cong \mathbb{C}\sigma_X$, where σ_X is a nowhere degenerate holomorphic 2-form. For a hyper-Kähler manifold, the group $\mathrm{H}^2(X, \mathbb{Z})$ has a canonical lattice structure; more precisely it is a free $\mathbb{Z}$-module with an integral bilinear form of signature $(3, b_2(X) - 3)$. The Néron–Severi group (or lattice) of X is defined as

$$\mathrm{NS}(X) = \mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X) \subset \mathrm{H}^2(X, \mathbb{C}).$$

The transcendental lattice $\mathrm{T}(X)$ is its orthogonal complement with respect to the bilinear form. The following is an easy way to prove corollary of [5, Proposition 4.1].

Proposition 1.1. — *Let X be a projective hyper-Kähler manifold. A group of automorphisms $G \subset \mathrm{Aut}(X)$ is finite if and only if it fixes an ample class on X.*

For the general theory of hyper-Kähler manifolds, we refer to [5], for example.

1.2. EPW-sextics, their double covers and automorphisms. — EPW-sextics are singular sextic hypersurfaces in $\mathbb{P}^5$, first constructed by Eisenbud, Popescu and Walter ([10]). O'Grady showed in [19] that they have a natural double cover, which is, in general, a smooth hyper-Kähler fourfold of type $\mathrm{K3}^{[2]}$. We briefly recall the construction.

Fix a 6-dimensional $\mathbb{C}$-vector space V_6 and choose an isomorphism $\bigwedge^6 V_6 \cong \mathbb{C}$. It induces a symplectic form $(\alpha, \beta) := \alpha \wedge \beta$ on $\bigwedge^3 V_6 \cong \mathbb{C}^{20}$. Denote by $\mathbb{LG}(\bigwedge^3 V_6)$ the Grassmannian of Lagrangian subspaces. Consider the vector subbundle $F \subset \bigwedge^3 V_6 \otimes \mathcal{O}_{\mathbb{P}(V_6)}$, whose fiber over $[v] \in \mathbb{P}(V_6)$ is given by:

$$F_v = \{\alpha \in \bigwedge^3 V_6 \mid \alpha \wedge v = 0\}.$$

To an element $A \in \mathbb{LG}(\bigwedge^3 V_6)$ one associates the loci

$$Y_A[k] = \{[v] \in \mathbb{P}(V_6) \mid \dim(A \cap F_v) \geq k\},$$

and $Y_A = Y_A[1]$ is called an *EPW-sextic.* We define two divisors:

$$\Sigma = \{A \in \mathbb{LG}(\bigwedge^3 V_6) \mid \mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \emptyset\}$$
$$\Delta = \{A \in \mathbb{LG}(\bigwedge^3 V_6) \mid Y_A[3] = \emptyset\}.$$

For $A \in \mathbb{LG}(\bigwedge^3 V_6) \smallsetminus (\Sigma \cup \Delta)$ there is a canonical double cover $\pi_A \colon \widetilde{Y}_A \to Y_A$ branched along the surface $Y_A[2]$, where $\widetilde{Y}_A$ is a smooth hyper-Kähler fourfold of type $\mathrm{K3}^{[2]}$ called *double EPW-sextic*, originally defined in [19]. Moreover, $\widetilde{Y}_A$ carries a canonical polarization $H = \pi_A^* \mathcal{O}_{Y_A}(1)$, and the image of the morphism $\widetilde{Y}_A \to \mathbb{P}(\mathrm{H}^0(\widetilde{Y}_A, H)^\vee)$ is isomorphic to Y_A.

Recall that if $A \notin \Sigma$ the automorphisms of the sextics are all linear:

$$\mathrm{Aut}(Y_A) = \{g \in \mathrm{PGL}(V_6) \mid (\bigwedge^3 g)(A) = A\}, \tag{1}$$

and this is a finite group by [4, Proposition B.9].

Consider the embedding $\mathrm{Aut}(Y_A) \hookrightarrow \mathrm{PGL}(V_6)$ and let G be the inverse image of $\mathrm{Aut}(Y_A)$ via the canonical map $\mathrm{SL}(V_6) \to \mathrm{PGL}(V_6)$. Then, G is an extension of $\mathrm{Aut}(Y_A)$ by the cyclic group $\langle\gamma\rangle$ of order 6, so we have an induced representation of G on $\bigwedge^3 V_6$, and this factors through a representation of $\widetilde{\mathrm{Aut}}(Y_A) := G/\langle\gamma^2\rangle$. Since A is preserved by this action, we have a morphism of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle\gamma^3\rangle & \longrightarrow & \widetilde{\mathrm{Aut}}(Y_A) & \longrightarrow & \mathrm{Aut}(Y_A) & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{GL}(A) & \longrightarrow & \mathrm{PGL}(A) & \longrightarrow & 1 \end{array} \tag{2}$$

and by [8, Lemma A.1] the vertical maps are injective.

Denote by $\mathrm{Aut}_H(\widetilde{Y}_A)$ the group of automorphisms fixing the polarization H.

Proposition 1.2 (Kuznetsov, [8, Proposition A.2]). — *Let $A \notin \Sigma$ be a Lagrangian space. There is an isomorphism*

$$\mathrm{Aut}_H(\widetilde{Y}_A) \cong \mathrm{Aut}(Y_A) \times \langle\iota\rangle$$

where ι is the anti-symplectic involution associated to the double covering π_A and the factor $\mathrm{Aut}(Y_A)$ corresponds to the subgroup of symplectic automorphisms.

When $A \notin \Sigma$, there is a canonical connected double covering [6, Theorem 5.2(2)]

$$\widetilde{Y}_A[2] \to Y_A[2]$$

and by [8, Proposition A.6 (Kuznetsov)] we have an injection $\widetilde{\mathrm{Aut}}(Y_A) \hookrightarrow \mathrm{Aut}(\widetilde{Y}_A[2])$.

Recall that the analytic representation of a finite group G acting on an Abelian variety X is the composition

$$G \to \mathrm{End}_{\mathbb{Q}}(X) \to \mathrm{End}_{\mathbb{C}}(T_{X,0}). \tag{3}$$

We recall the following useful result, where $\mathrm{Alb}(\widetilde{Y}_A[2])$ denotes the Albanese variety of $\widetilde{Y}_A[2]$.

Proposition 1.3 ([8, Proposition A.7]). — *Suppose that $Y_A[3] = \emptyset$. The restriction to the subgroup $\widetilde{\mathrm{Aut}}(Y_A)$ of the analytic representation of $\mathrm{Aut}(\widetilde{Y}_A[2])$ on $\mathrm{Alb}(\widetilde{Y}_A[2])$ is the injective middle vertical map in the diagram (2).*

1.3. GM varieties and their automorphisms. — Let V_5 be a 5-dimensional complex vector space. A GM variety (shorthand for ordinary Gushel–Mukai) of dimension $n \in \{3,4,5\}$ is the smooth complete intersection of the Grassmannian $\mathrm{Gr}(2, V_5) \subset \mathbb{P}(\bigwedge^2 V_5)$ with a linear space $\mathbb{P}^{n+4}$ and a quadric. These varieties are Fano varieties with Picard number 1, index $n-2$ and degree 10.

There is a bijection between the set of isomorphism classes of GM varieties of dimension n and isomorphism classes of triples (V_6, V_5, A), where $A \in \mathbb{LG}(\bigwedge^3 V_6) \smallsetminus \Sigma$ and $V_5 \subset V_6$ is a hyperplane that satisfies

$$\dim(A \cap \bigwedge^3 V_5) = 5 - n.$$

The correspondence is outlined in [4, Theorem 3.10 and Proposition 3.13(c)].

In particular, for $A \in \mathbb{LG}(\bigwedge^3 V_6) \smallsetminus (\Sigma \cup \Delta)$ there is an associated double EPW-sextic and associated families of GM varieties of dimensions $3, 4$ and 5.

2. EPW-sextics with an action of $\mathcal{A}_7$

The general idea is to find a Lagrangian $A \subset \bigwedge^3 V_6$ that is invariant under the action of the group $\mathcal{A}_7$ to get an invariant EPW-sextic. According to [3], there exists a group denoted by $3.\mathcal{A}_7$ with η an element of order 3 such that $3.\mathcal{A}_7/\langle\eta\rangle \cong \mathcal{A}_7$. This group has a unique irreducible representation $\rho\colon 3.\mathcal{A}_7 \to \mathbb{C}^6 \cong V_6$ and η acts by multiplication for a third root of unity. Generators for this representation can be found in [22]. Notice that this induces a representation on $\bigwedge^3 V_6$, which descends to a representation of the quotient $\mathcal{A}_7$ on $\bigwedge^3 V_6$. Denote this (faithful) representation of $\mathcal{A}_7$ by W.

The irreducible complex representations of $\mathcal{A}_7$ of dimension smaller than or equal to 20 have dimensions $1, 6, 10, 10, 14, 14$ and 15 and are described in Table 2.1.

Lemma 2.1. — *The representation W decomposes as the direct sum $W = A_1 \oplus A_2$ of the only two irreducible 10-dimensional representations $R_1 = (A_1, \rho_1) \cong W_{10}$ and $R_2 = (A_2, \rho_2) \cong W'_{10}$ of the group $\mathcal{A}_7$. Moreover, the underlying vector spaces $A_1, A_2 \subset \bigwedge^3 V_6$ of those representations are Lagrangian.*

Proof. — The fact that W has the mentioned decomposition is just a computation of characters (see Table 2.1). The fact that the subrepresentations are Lagrangian is easily checked with computer algebra after determining the